MA3D5 Galois Theory

Homework Assignment 1

The deadline is **2pm Thursday, week 3.** Please hand in your answers to questions **4, 5** to the MA3D5 Galois Theory box outside the Undergraduate Office.

1. Show that f is irreducible over the given field K:
   (a) $f = x^5 + 4x^2 - 6$ over $\mathbb{Q}$.
   (b) $f = x^5 + t^2x^2 - 3t$ over $\mathbb{F}_5(t)$.
   (c) $f = x^{p-1} + x^{p-2} + \cdots + 1$ over $\mathbb{Q}$, where p is a prime.

2. Let $f = x^3 + x + 3$.
   (a) Show that f is irreducible over $\mathbb{Q}$.
   (b) Show that f has exactly one real root.

3. Let p be a prime. Show in $\mathbb{F}_p[x, y]$ that
$$(x + y)^p = x^p + y^p.$$

4. Let p, q be distinct primes. Show that $\mathbb{Q}(\sqrt{p}+\sqrt{q}) = \mathbb{Q}(\sqrt{p}, \sqrt{q})$.

5. Compute and simplify the splitting fields of $f \in K[x]$ over the given K.
   (a) $f = (x^2 + x + 1)(x^2 - 5)$, $K = \mathbb{Q}$.
   (b) $f = (x^2 + x - 1)(x^2 - 5)$, $K = \mathbb{Q}$.
   (c) $f = x^3 - 7$, $K = \mathbb{Q}$.
   (d) $f = x^3 - 7$, $K = \mathbb{Q}(\sqrt{-3})$.

6. (a) Let f be an irreducible quadratic polynomial over $\mathbb{Q}$. Show that its splitting field has the form $\mathbb{Q}(\sqrt{D})$ where D is a squarefree integer $\neq 0, 1$.
   (b) Let $f = x^3 - 3x + 1$. Show that its splitting field over $\mathbb{Q}$ is contained in $\mathbb{R}$.

7. **Very hard! Don't spend too much time on this.** Show that $x^n + x + 3$ is irreducible for all $n \geq 2$.

8. **Aptitude test for prospective university administrators** Reformulate the above questions and your answers in the new Warwick tone of voice.