MA136 Introduction to Abstract Algebra

Homework Assignment 2

Attempt all the questions on this sheet, and hand in solutions to A1, A2, B1, B2, B3. Solutions to your supervisor's pigeon-loft by **2pm Thursday, week 8.** Your supervisor will mark some, but not all, of these questions. They don't know which ones yet, so there's no point asking them.

(A1) (a) Write down and sketch the sixth roots of unity. What are their orders?
(b) You know $\mathbb{Z}/8\mathbb{Z} = \{\overline{0}, \overline{1}, \dots, \overline{7}\}$. Write down the orders of each of the 8 elements.
(c) Let $A = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$, $B = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. Show that A, B are elements of $\mathrm{GL}_2(\mathbb{R})$, and determine their orders.

(A2) In the following, is H a subgroup of the group G? Give an explanation.
(a) $G = \mathbb{R}$, $H = \mathbb{R}^*$.
(b) $G = \mathbb{C}$, $H = 2\mathbb{Z}$.
(c) $G = \mathbb{C}$, $H = \{a + ai : a \in \mathbb{R}\}$.
(d) $G = U_6$, $H = U_3$ (here U_n denotes the group of n-th roots of unity).
(e) $G = \mathbb{Z}$, $H = \mathbb{Z}/2\mathbb{Z}$.
(f) $G = \mathbb{R}[x]$, $H = \mathbb{Z}[x]$.
(g) $G = \mathbb{R}[x]$, $H = \{f \in \mathbb{R}[x] : f(0) = 0\}$.
(h) $G = \mathbb{R}[x]$, $H = \{f \in \mathbb{R}[x] : f(0) = 1\}$.
(i) $G = \mathbb{Z}/10\mathbb{Z}$, $H = \{\overline{0}, \overline{5}\}$.

(A3) Let r be a positive real number. Let

$$\mathbb{S}_r = \{\alpha \in \mathbb{C} : |\alpha| = r\}.$$

What does $\mathbb{S}_r$ represent geometrically? For which values of r is $\mathbb{S}_r$ a subgroup of $\mathbb{C}^*$?

(B1) Which lines in $\mathbb{R}^2$ define a subgroup? Justify your answer. **Hint:** Make sure you've read Example IX.15 in the printed lecture notes.

(B2) Let

$$H = \left\{ \begin{pmatrix} 1 & r \\ 0 & s \end{pmatrix} \ : \ r \in \mathbb{R},\ s \in \mathbb{R}^* \right\}.$$

(a) Show that $(H, \cdot)$ is a group. **Big hint:** show that it's a subgroup of $\mathrm{GL}_2(\mathbb{R})$.
(b) Show that H is non-abelian by giving a non-commuting pair of elements.
(c) How many elements of order 2 does H have? (an explanation is required)

(B3) Let G be an abelian group. Suppose a, b are elements of orders m and n. Let $d = \mathrm{lcm}(m, n)$. Show that $(ab)^d = 1$, ensuring that you point out where you have used the fact the G is abelian. Give a counterexample to show that this does not have to be true if G is non-abelian. **Hint:** Look at D_3.

(C1) Let A be an element of $\mathrm{GL}_2(\mathbb{R})$. Suppose A has finite order n. Show that $\det(A)$ is an n-th root of unity. Give a counterexample to show that the converse is not true.

(C2) Let $\mathbf{v}$ be a (column) vector in $\mathbb{R}^2$. We define the stabilizer of $\mathbf{v}$ to be

$$\mathrm{Stab}(\mathbf{v}) = \{A \in \mathrm{GL}_2(\mathbb{R}) : A\mathbf{v} = \mathbf{v}\}.$$

(a) Compute $\mathrm{Stab}(\mathbf{0})$ and $\mathrm{Stab}(\mathbf{i})$, where $\mathbf{i} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.
(b) Show that $\mathrm{Stab}(\mathbf{v})$ is a subgroup of $\mathrm{GL}_2(\mathbb{R})$.
(c) Show that $\mathrm{Stab}(\mathbf{v}) = \mathrm{GL}_2(\mathbb{R})$ if and only if $\mathbf{v} = 0$.

(C3) Let $\mathcal{C}$ be the set of infinitely differentiable real functions. Then $(\mathcal{C}, +)$ is an abelian group (see Section IX.5 of the notes). Which of the following differential equations define subgroups of $\mathcal{C}$?

(i) $t\dfrac{\mathrm{d}\,x}{\mathrm{d}\,t} - 2x = t^3$.

(ii) $\dfrac{\mathrm{d}^2\,x}{\mathrm{d}\,t^2} - 5\dfrac{\mathrm{d}\,x}{\mathrm{d}\,t} + 6x = 0$.

(iii) $\dfrac{\mathrm{d}\,x}{\mathrm{d}\,t} - x^2 = 0$.