# Limits of Permutations

$S_n = \{\text{bijections } [n] \to [n]\}$

**Sampling** from $\Pi \in S_n$:

- pick random $X = \{x_1 < \dots < x_k\} \subseteq [n]$
- output $\sigma(\pi, \Pi) \in S_k$
  $\quad = \Pi|_X$
  $\sigma(i) < \sigma(j)$ iff $\Pi(x_i) < \Pi(x_j)$

Eg: $\Pi = 4\underline{5}\underline{3}21\underline{6}$
$X = \{2, 3, 6\}$
$\Pi|_X = 213$

**Density** of $\sigma$ in $\Pi$:

$$t(\sigma, \Pi) = \frac{|\{X : \Pi|_X = \sigma\}|}{\binom{n}{k}} = \Pr(\sigma(X, \Pi) = \sigma)$$

Eg $t(213, 453216) = \frac{9}{\binom{6}{3}}$

Def $(\Pi_n)$ **converges** if

$|\Pi_n| \to \infty$ & $\forall \sigma\ t(\sigma, \Pi_n) \to \phi(\sigma)$

One point of view $\phi \in [0,1]^{\bigcup_n S_n}$

$Z = \{\text{measures } \mu \text{ on } ([0,1]^2, \mathcal{B}) \text{ with } \underline{\text{uniform marginals}}\}$

$\forall$ Borel $A$ $\quad \mu(A \times [0,1]) = \mu([0,1] \times A) = \lambda(A)$.

Examples: • $\mu = \lambda$

• $\mathrm{supp}(\mu) = \text{diagonal}$
$\mu(A) = \lambda(P_1(A))$, $A \subseteq \text{diagonal}$

•

Sampling from $\mu \in \mathcal{Z}$:

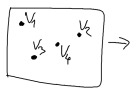
- independent $(X_i, Y_i) \sim \mu$, $i \in [k]$.
- Prob(some $X_i = X_j$) $= 0$
- Pick (unique) $\pi \in S_k$ s.t.

$$X_{\pi^{-1}(1)} < X_{\pi^{-1}(2)} < \dots < X_{\pi^{-1}(k)}$$

$(\Leftrightarrow \pi(i) < \pi(j)$ iff $X_i < X_j)$

- output: profile of $Y_{\pi^{-1}(1)}, \dots, Y_{\pi^{-1}(k)}$ $\quad \sigma(k, \mu)$

Eg:




$\to 4123$

- $\sigma(k, \text{Lebesgue})$ is a uniform distribution on $S_k$
- $\sigma(k, \text{diagonal})$ is concentrated on the identity $\in S_k$

For $\pi, \tau \in S_k$, let $A_{\pi,\tau} \subseteq [0,1]^{2k}$ be the event that

$$\left(X_{\pi^{-1}(1)} < \dots < X_{\pi^{-1}(k)}\right) \& \left(Y_{\tau^{-1}(1)} < \dots < Y_{\tau^{-1}(k)}\right)$$

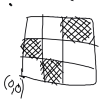
If we land in $A_{\pi,\tau}$, then output $\tau \circ \pi^{-1}$.

Def: density of $\sigma \in S_k$ in $\mu \in \mathcal{Z}$

$$t(\sigma, \mu) = \Pr(\sigma(k, \mu) = \sigma)$$
$$= \sum_{\rho \in S_k} \mu^k(A_{\rho, \sigma\rho})$$
$$= k! \, A_{\rho, \sigma\rho}, \text{ some (any) } \rho \in S_k$$

For $\Pi \in S_n$, let $\mu_\Pi \in \mathcal{Z}$ be concentrated on $\left[\frac{i-1}{n}, \frac{i}{n}\right] \times \left[\frac{\Pi(i)-1}{n}, \frac{\Pi(i)}{n}\right)$ (uniform on these $n$ squares)

E.g. $\Pi = 213$.



Sample $G(\kappa, \mu_\Pi)$:

- $\Pr(\text{two points end in the same } \frac{1}{n}\times\frac{1}{n} \text{ square}) \le \frac{\binom{\kappa}{2}}{n}$
- So $\left| t(G, \underset{\uparrow}{P}, \Pi) - t(G, \mu_\Pi) \right| \le \frac{\binom{\kappa}{2}}{n}$
  $P_\kappa$

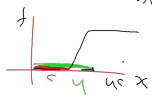
**Theorem 29** (Floyen et al '12)
$\forall$ conv. $(\Pi_n)$ $\exists \mu \in \mathcal{Z}$ s.t.
$\forall G$ $\lim t(G, \Pi_n) \to t(G, \mu)$.
Moreover, this $\mu$ is unique.

**Proof.** Prokhorov's Thm (1950's):
$\exists$ subsequence $(\mu_{\Pi_{n_i}})$ that weakly converges to $\mu$.
WLOG, $\mu_n = \mu_{\Pi_n} \to \mu$ weakly
(i.e. $\forall$ cts $f: [0,1]^2 \to \mathbb{R}$
$\int f(x)\, d\mu_n(x) \to \int f(x)\, d\mu(x)$).

We have $\mu \in \mathcal{Z}$. Indeed,
Let $A \subseteq [0,1]^2$ be Borel.
By regularity of Lebesgue measure $\lambda$ $\forall \varepsilon > 0$ $\exists$ closed $C \subseteq A$, $\exists$ open $U \supseteq A$ s.t. $\lambda(U \setminus C) < \varepsilon$.




$\exists$ cts fn $f: [0,1]^2 \to [0,1]$
s.t. $f = 0$ on $C$ & $f = 1$ on $U$
(e.g. $f(x) := \frac{\operatorname{dist}(C, U^c \cup x)}{\operatorname{dist}(C, U^c)}$)




$g(x,y) = f \circ P_x$ (i.e. $g(x,y) = f(x)$)
$\int (1-g)\, d\mu_n = \int (1-f)\, dx = \lambda(A) \pm \varepsilon$

But $\int (1-g)\, d\mu_n \to \int (1-g)\, d\mu$.

So $\mu \in \mathcal{Z}$.

To show $\lim t(\sigma, \mu_n) = t(\sigma, \mu)$ it is enough to show $\forall \pi, \tau \in S_k$

$$\mu_n^k(A_{\pi,\tau}) \to \mu^k(A_{\pi,\tau})$$

Boundary of $A_{\pi,\tau}$ is "nice".

Namely, $\Pr_{\mu^2}(|X_1 - X_2| < \varepsilon) < 2\varepsilon$

So $\mu^k(\partial A_{\pi,\tau}) = 0$, so

$A_{\pi,\tau}$ is a set of continuity for $\mu^k$. Thus $\mu_n^k(A_{\pi,\tau}) \to \mu^k(A_{\pi,\rho})$

(Since $\mu_n^k \to \mu^k$ weakly).

This shows existence of $\mu$.

Uniqueness: suppose we have $\mu$ & $\mu'$.

To show $\mu = \mu'$, enough to show $\forall$ cts. fn $f: [0,1]^2 \to \mathbb{R}$ that

$$\int f(x,y)\, d\mu(x,y) = \int f(x,y)\, d\mu'(x,y). \quad (*)$$

By Weierstrass Thm, enough to show $(*)$ for polynomials.

$$\int x^k\, d\mu(x,y) = \int x^k\, dx = \frac{1}{k+1}$$

$$\int xy\, d\mu(x,y) = \int X_1 Y_1\, d\mu(X_1, Y_1)$$

$$= \int \Pr(X_2 < X_1) \Pr(Y_3 < Y_1)\, d\mu^3$$

$$= \int_{X_2 < X_1,\, Y_3 < Y_1} d\mu^3 = \sum_{\substack{\pi(2) < \pi(1) \\ \pi(3) < \pi(1)}} \mu^3(A_{\pi,\tau}) =$$

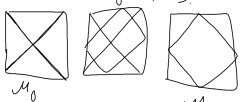
$$\int xy\, d\mu'(x,y) = \ldots = \sum_{\substack{\pi(2) < \pi(1) \\ \tau(3) < \tau(1)}} \underbrace{(\mu')^3(A_{\pi,\tau})}_{= \frac{t(\tau \circ \pi^{-1}, \mu')}{3!}}$$

Since $t(\cdot, \mu) = t(\cdot, \mu')$, these sums are the same & $\mu = \mu'$. $\square$

**Theorem 30** (Hoppen et al ≥13)
Let $\mu \in \mathcal{Z}$. Let $\Pi_n \sim \mathcal{U}(n, \mu)$, $n \in \mathbb{N}$, independently. Then
$$\Pr\left((\Pi_n) \to \mu\right) = 1.$$

**Proof:** Fix $\sigma \in S_k$.
Let $X = t(\sigma, \Pi_n)$.
One can show $\mathbb{E}(X) = t(\sigma, \mu)$.
Also, $X$ depends on $n$ indep. R.V.
$(x_i, y_i) \sim \mu$, & one change in $X$ is $\leq \binom{n-1}{k-1} / \binom{n}{k} = \frac{k}{n}$.
Azuma's Inequality:
$$\Pr\left(|X - EX| > \varepsilon\right) \leq e^{-\frac{\varepsilon^2}{2n(k/n)^2}}$$
$$= e^{-\varepsilon^2 n / 2k^2}.$$
Apply Borel-Cantelli. □

**Application:**
Call $\mu \in \mathcal{Z}$ $k$-symmetric if $\forall \sigma \in S_k \; t(\sigma, \mu) = \frac{1}{k!}$.
For example, $\lambda \in \mathcal{Z}$ is $k$-symmetric $\forall k$.
Qn (Graham): $\exists k \; \forall \mu \in \mathcal{Z}$ $\mu$ is $k$-symmetric $\Rightarrow \mu = \lambda$?
Ex not true for $k = 3$.




$t(123, \mu_0) = \frac{1}{4} > \frac{1}{6}$ $t(123, \mu_1) = \frac{1}{8} < \frac{1}{6}$
$\exists \mu_t$ s.t. $t(123, \mu_t) = \frac{1}{6}$
symmetries $\Rightarrow$ 123, 231, 312 — same dens.
So $\forall \sigma \in S_3, t(\sigma, \mu_t) = \frac{1}{6}$ 132, ... — same dens.

**Thm** $\mu$ is $S$-symmetric $\Rightarrow \mu = \lambda$.

**Proof** Def $F(a,b) := \mu([0,a)\times[0,b))$

(e.g. if $\mu=\lambda$, then $F(a,b)=ab$).

Cauchy-Schwartz:

$$\left(\int F(x,y)\,xy\,dx\,dy\right)^2 \le \left(\int F^2(x,y)\,dx\,dy\right)\left(\int x^2y^2\,dx\,dy\right)$$

$\int F(x,y)\,xy\,dx\,dy = \frac{1}{9}$ ?

$\int F^2(x,y)\,dx\,dy = \frac{1}{9}$ ?

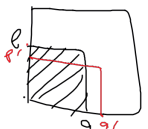
$\int x^2y^2\,dx\,dy = \frac{1}{9}$

Equality in CS $\Rightarrow \exists C$ s.t.

$F(a,b) = Cab$ a.e. $(a,b)\in[0,1]^2$

But $F(a,b)$ is a cts. function.

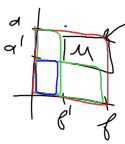
(namely, $|F(a',b')-F(a,b)| \le |a-a'|+|b-b'|$)




So $F(a,b) = Cab$ $\forall a,b$.

Also, $C=1$ (look at $(a,b)=(1,1)$).

Since




$F(a,b)-F(a',b)-$
$-F(a,b')+F(a',b')$