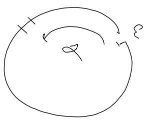
Def $m(\mathcal{G}) = \sup\{\mu(V(M)) :$ Borel matching M in $\mathcal{G}\}$

Ex $m(R_\alpha) = 1$



Thm 10.3 (Lyons-Nazarov '10)
Bipartite $\mathcal{G} = (A \sqcup B, \mathcal{B}, E, \mu)$ s.t. $\mu(A) = \mu(B) = 1/2$ & expansion
($\exists\, \varepsilon > 0\ \forall$ Borel X in a part
$\mu(N(X)) \geq (1+\varepsilon)\mu(X)$ or $> \frac{1}{4} = \frac{1}{2}\mu(A)$)
then $m(\mathcal{G}) = 1$ (more strongly, $\exists$ Borel matching M s.t. $\mu(V(M)^c) = 0$)

Pf Lem 10.1 gives $M_1, M_2, M_3, \dots$ (Borel matchings)

Def $M = \bigcup_j \bigcap_{i \geq j} M_i$, Borel matching

μ invariant & M Borel $\Rightarrow$
$\mu(A \cap V(M)) = \mu(B \cap V(M))$

Enough

(i) $\mu(A \setminus V(M_i)) \to 0$ as $i \to \infty$

(ii) $\sum_{i=1}^{\infty} \delta_i < \infty$, $\delta_i = \mu\{a \in A : M_i(a) \neq M_{i+1}(a)\}$

Indeed then $\forall j$

$$\mu(A \setminus V(M)) \leq \underbrace{\mu(A \setminus V(M_j))}_{\to 0 \text{ by (i)}} + \underbrace{\sum_{i=j}^{\infty} \delta_i}_{\to 0 \text{ by (ii)}}$$

So $\mu(A \setminus V(M)) = 0$ ☺

Proving (i) & (ii)

<u>Recall</u> M_i has no aug path of length $\leq 2i-1$

$A_0 = A \setminus V(M_i)$

[expansion $\Rightarrow \mu(A_0) \leq (1+c)^{-i}$

$\Rightarrow \quad \delta_i \leq i\,\mu(A_0) \leq i(1+c)^{-i}$, summable

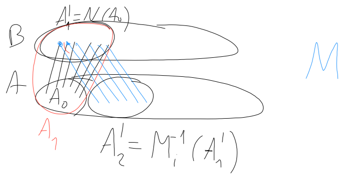
$\Rightarrow$ (i) & (ii)]

$A_0 = A \setminus V(M_i)$, $B_0 = B \setminus V(M_i)$

For $j \geq 1$: $A_j = \{$vertices reached from A_0 by alternating path of length $\leq j\}$

$B_j = \{$ ——— , ——— B_0 ——— / ——— $\}$.

$A'_j = A_j \setminus A_{j-1}$, $B'_j = B_j \setminus B_{j-1}$



Claim 1 $\forall j \leq i-1$ $A_{2j+1} \cap B = N(A_{2j} \cap A)$ □

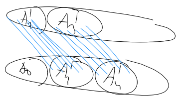
Claim 2 $\forall j \leq i$ M_i^{-1} gives a bijection from A'_{2j-1} to A'_{2j}

So $\forall j \leq i-1$:

$\mu(A_{2j+1} \cap B) \overset{cl1}{=} \mu(N(A_{2j} \cap A))$

$\geq (1+\varepsilon)\mu(A_2 \cap A)$ or $> 1/4$

$\mu(A_{2j} \cap A) \overset{cl2}{=} \mu(A_{2j-1} \cap B)$

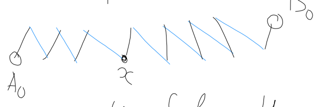


$\Rightarrow \forall j \leq i-1$ $\mu(A_j) \geq (1+\varepsilon)^{j/2}\mu(A_0)$ or $> \frac{1}{2}$

Same for B side

Claim 3 $A_{i-1} \cap B_i = \emptyset$

Pf



$\Rightarrow$ aug. path of length $\leq$

$\leq i-1+i = 2i-1 \Rightarrow\Leftarrow$

WLOG $\mu(A_{i-1}) \leq \frac{1}{2}$

So $(1+\varepsilon)^{\frac{i-1}{2}}\mu(A_0) \leq \frac{1}{2}$ □

11 Measurable equidecompositions

(S^2, μ), $\mu(S^2)=1$,
μ is $SO(3)$-invariant

Thm 11.1 (Banach-Tarski '24)
$\forall A, B \subseteq S^2$ s.t. $\exists$ finite $T \subseteq SO(3)$
with $T.A = \bigcup_{\gamma \in T} \gamma A$ & $T.B$ being S^2
Then $A \sim B$ using $SO(3)$

Thm 11.2 (Grabowski-Máthé-P '16)
If additionally, $A, B \in \mathcal{B}_\mu$
& $\mu(A)=\mu(B)$ then all parts can be in $\mathcal{B}_\mu$

Pf Fix finite $S \subseteq SO(3)$
Let $\mathcal{L}^2 = \mathcal{L}^2(S^2, \mu) = \{$ measurable $f: S^2 \to \mathbb{R}$ s.t. $\|f\|_2 = \sqrt{\int f^2 d\mu} < \infty \}$

Def $T_S : \mathcal{L}^2 \to \mathcal{L}^2$ by
$$(T_S f)(x) = \frac{1}{|S|} \sum_{\gamma \in S} f(\gamma^{-1} x), \quad f \in \mathcal{L}^2, x \in S^2$$
Then $\forall f$ $\|T_S f\|_2 \leq \|f\|_2$

S has spectral gap if $\exists \varepsilon > 0$
s.t. $\forall f \in \mathcal{L}^2 \left(\int f d\mu = 0 \Rightarrow \|T_S f\|_2 \leq (1-\varepsilon)\|f\|_2 \right)$

Drinfeld '84 : $\exists$ finite $S \subseteq SO(3)$

Fix such S. Assume $S = S^{-1}$

$\mathcal{G} = (S^2, \mathcal{B}, \mathrm{graph}(S), \mu)$

Then $\mathcal{G}$ is a (non-bip) expander

Take $Y \in \mathcal{B}$, $y = \mu(Y) \in (0,1)$

Def $f = (1-y)\mathbb{1}_Y - y\,\mathbb{1}_{Y^c}$

$\int f \, d\mu = 0$

So $\|T_S f\|_2 \leq (1-\varepsilon)\|f\|_2$

NB $\underset{\substack{\| \\ 0}}{\int T_S f \, d\mu} = \sum_{\gamma \in S} \frac{1}{|S|} \underbrace{\int f(\underbrace{\gamma^{-1}x}_{y}) \, d\mu(x)}_{=0}$

<u>Calculations</u>: $\mu((SY)\setminus Y) \geq \frac{\varepsilon y(1-y)}{|S|}$

How to make $\mathcal{G}[A,B]$ Bip expander?

Recall $T.\mathcal{B} = S^2$, $T = T^{-1}$

Let $t = |T|$

If $\mu(Y) = y \leq 1 - \frac{1}{2t}$, then

$\mu((S.Y)\cancel{\setminus Y}) \geq \left(1 + \frac{\varepsilon}{|S| \cdot 2t}\right)\mu(Y)$

(assume $S \ni e$)

So pick $\ell \in \mathbb{N}$ s.t. $\left(1 + \frac{\varepsilon}{|S|2t}\right)^{\ell} > 2t$

Then $\mu(S^{\ell} Y) \geq 2t\mu(Y)$ or $> 1 - \frac{1}{2t}$

So $\exists \gamma \in T$ s.t. $\mu((S^{\ell} Y) \cap \gamma.\mathcal{B}) \geq 2\mu(Y)$

$\mu((\gamma^{-1} S^{\ell} Y) \cap \mathcal{B})$ // invariance of μ or $> \frac{1}{2}\mu(\mathcal{B})$

So TS makes $[A,B]$ Bip expander

Lyons-Nazarov (Thm 10.2) $\Longrightarrow$

$\mathcal{G}(A \cup B, \mathcal{B}, \mathrm{graph}(TS), \mu)$ has

Borel M s.t. $\mu(\underbrace{A \setminus V(M)}_{N}) = 0 = \mu(\underbrace{B \setminus V(M)}_{N'})$

N, M are Borel

Apply Banach-Tarski to

$[N \cup M] = \langle TS \rangle (N \cup M)$

(assuming T was enlarged to have all elements required by BT)

$\square$