A: set, $W \subseteq A^{\mathbb{N}}$

Game $G(W)$:

I	a_1		a_3		
II		a_2		a_4	$\rightsquigarrow a = (a_1, a_2, \dots)$

I wins if $a \in W$ (o/w II wins)

A: discrete top. space ($\tau = 2^A$)

Product top on $A^{\mathbb{N}}$

Thm 5.2 (Gale-Stewart)
$W \subseteq A^{\mathbb{N}}$ is open or closed $\Rightarrow$ $G(W)$ is determined

Pf Players O & C (w. sets W_O, open, & $W_C = A^{\mathbb{N}} \setminus W_O$, closed))

Def $p \in A^{<\infty}$ is winning for O if O has a winning strategy from p

Claim If p is not winning for O, then
C moves $\Rightarrow \exists a$ st $p^\frown a$ is not winning for O
O moves $\Rightarrow \forall a$ $p^\frown a$ is not winning for O

If $p = ()$ is winning for O, done ✓
Suppose not
Strategy for C: play st the new p is not winning for O

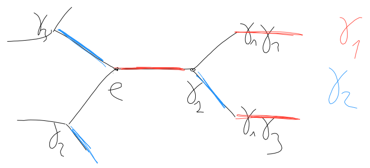
Resuting $a = (a_1, a_2, a_3, \ldots)$
Claim $a \in W_C$
O/w: $\exists n \ \forall q \ (a_1, a_2, \ldots, a_n)^\frown q \in W_O$
But then $(a_1, \ldots, a_n)$ is winning for O
$\Rightarrow\Leftarrow$ □

Thm 5.3 (Martin'75) $\forall$ Borel $W \subseteq A^{\mathbb{N}}$ is determined $\square$

Thm 4.6a (Marks'16) $\forall n \geq 3$

$\exists$ n-regular acyclic Borel G s.t. $\chi_B(G) = n+1$

Pf $\Gamma = \underbrace{\mathbb{Z}_2 * \dots * \mathbb{Z}_2}_{n} = \langle \gamma_1, \dots, \gamma_n \mid \gamma_i^2 = e \rangle$



$V = \mathbb{N}^{\Gamma} = \{x : \Gamma \to \mathbb{N}\} = \{\text{labellings}\}$

$\Gamma \curvearrowright V$: $(\gamma . x)(\eta) = x(\gamma^{-1}\eta)$

$[\delta . (\gamma . x) = (\delta\gamma) . x]$

$G = \mathrm{Graph}(\gamma_1, \dots, \gamma_n)$

$E = \{\{x, \gamma_i . x\} : x \in V, i \in [n], x \neq \gamma_i . x\}$

Shift

$X = \{x \in V : \underbrace{\forall \alpha \in \Gamma\ \forall i\ \ x(\alpha) \neq x(\alpha\gamma_i)}_{(*)}\}$

Free Part $= \{x \in V : \forall \gamma \in \Gamma \setminus e\ \ \gamma . x \neq x\}$

$= \{x : \mathrm{Comp}(x) \text{ is n-reg tree}\}$

Ex $x \equiv \text{Const}$ is NOT

$x(\gamma) = |\gamma| \bmod 2$ is NOT

← reduc. word in γ_i's $\quad (\gamma_1\gamma_2 . x = x)$

reduced $\gamma_1\gamma_2\gamma_1$ ✓

$\gamma_1\gamma_1\gamma_3$ ✗

$\mathrm{Comp}(x) = (\{\gamma . x : \gamma \in \Gamma\}$, E restricted to this set$)$

Ex $x(\eta) = |\eta|$ is in Free part

If $\gamma . x = x \Rightarrow \underset{0}{x(e)} = (\gamma . x)(e) = x(\gamma^{-1}e) = |\gamma|$

$\Rightarrow \gamma = e$

$Y := X \cap \text{FreePart}$

$G \upharpoonright Y$: n-regular, acyclic Borel graph

aim $\chi_B(G \upharpoonright Y) > n$

Step A $\nexists$ Borel QC $c : X \to [n]$

Step B $\exists$ Borel QC $c : X \setminus Y \to [n]$

$\left[A \,\&\, B \Rightarrow \chi_B(G \setminus Y) > n \right]$

Suppose Borel $c : X \to [n]$ is QC

(ie A is false). Fix c

Recall $X = \{ x : \Gamma \to \mathbb{N} \text{ s.t. } (*) \}$

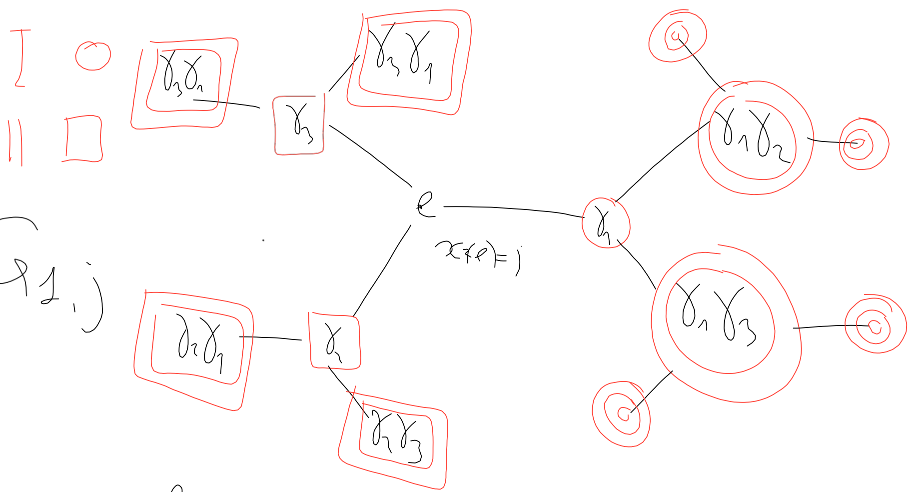
$\text{Game}_{i,j}$ players construct $x : \Gamma \to \mathbb{N}$

Let $x(e) = j$

Round n: I def x on $\gamma_i \gamma_{k_2} \cdots \gamma_{k_n}$

II $\gamma_s \gamma'_{k_2} \cdots \gamma'_{k_n}$ $(s \neq i)$

(reduced words)



$G_{1,j}$

$W_i = \{x \in X : c(x) \neq i\} \cup \{x \in X :$ II first violated $(*)\}$

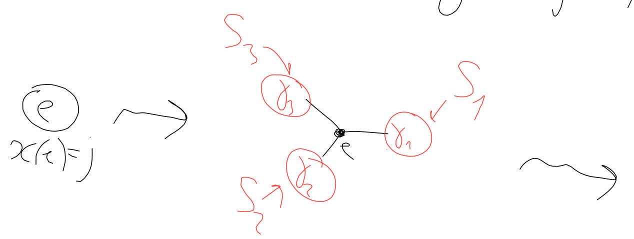
$(*)$ $\forall \alpha \in \Gamma \ \forall i \quad x(\alpha) \neq x(\alpha \gamma_i)$

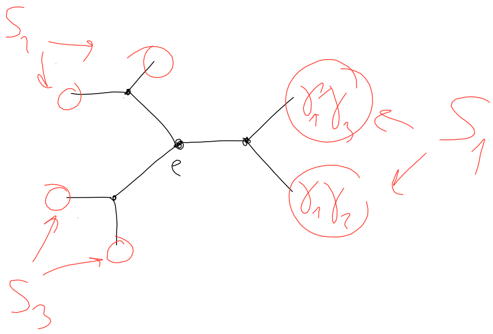
<u>Rule</u>: $x(\gamma_{k_1} \cdots \gamma_{k_m}) \neq x(\gamma_{k_1} \cdots \gamma_{k_{m-1}})$

c Borel $\Rightarrow W_i$ is Borel

$\forall j \ \exists i \in [n]$ II wins G_{ij}

[o/w: Thm 5.3 $\Rightarrow$ I has strategies $S_1, \dots, S_n$ winning $G_{1j}, \dots, G_{nj}$





$\rightsquigarrow$ gives $x \in X$

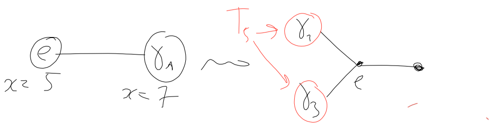
let $i = c(x)$ $\Rightarrow\Leftarrow$ S_i winning

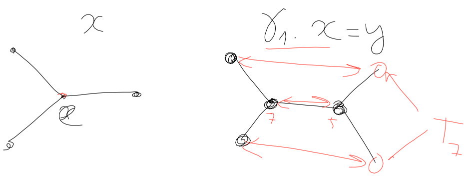
So $\forall j \ \exists i$ s.t. II can make $c(x) = i$ in G_{ij}

$\Rightarrow \exists\ j_0 \neq j_1$ st, some i,
II wins G_{i,j_0} & G_{i,j_1}, with strategies T_{j_0} & T_{j_1}

Eg II wins $G_{1,5}$ & $G_{1,7}$ with T_5 & T_7

Construct x:



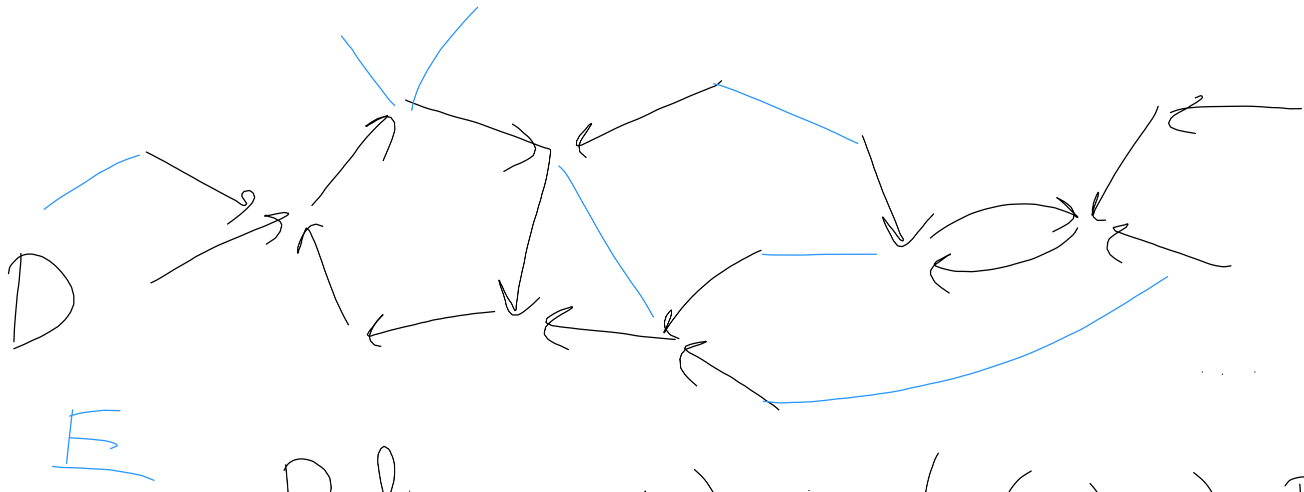


ete

$\Rightarrow x \in X$ $\quad c(x)=i$ & $c(\gamma_1 . x) = i$
(T_5 won) $\quad$ (T_7 won on $\gamma_1 x = y$)

$\Rightarrow\!\Leftarrow$ c is QC

<u>Step B</u> Find Borel digraph $D \subset E \restriction (X \setminus Y)$ st $\forall x$ out-degree is 1



Define $c(x)=i$ s.t. $(x, \gamma_i x) \in D$