Can you see this?

2

3

<u>Borel graph</u> is $(V, \mathcal{B}, E)$ st

- $(V, \mathcal{B})$ is SBS (standard Borel sp)
[i.e. $\exists$ Polish top τ st $\mathcal{B} = \sigma(\tau)$]
- $E \subseteq V \times V$ is Borel [NB: $\mathcal{B}(V) \times \mathcal{B}(V) = \mathcal{B}(V^2)$]
- $G = (V, E)$ is a graph (with $\Delta < \infty$)

<u>Lem 3.3</u> $(V, \mathcal{B})$ SBS, $A \subseteq V$ Borel $\Rightarrow$ $(A, \mathcal{B} \cap 2^A)$ is SBS. $\square$

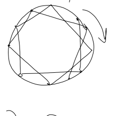
<u>Cor 3.4</u> $\forall A \subseteq V$ Borel,

$(V, \mathcal{B}, E) \restriction A = (A, \mathcal{B} \cap 2^A, E \cap A^2)$ is a Borel graph $\square$

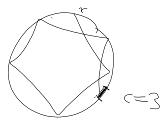
4 Borel chromatic numbers

Borel chr number $\chi_{\mathcal{B}}(G)$ $\stackrel{def}{=}$ is min $k \in \mathbb{N}$ st $\exists$ Borel vertex colouring $c: V \to [k] = \{1, \dots, k\}$ [ie $\exists$ Borel partition $V = V_1 \cup \dots \cup V_k$, with each V_i spanning no edges]

Ex V ctbl $\Rightarrow \chi_{\mathcal{B}}(G) = \chi(G)$

Ex R_α 

$\chi_{\mathcal{B}}(R_\alpha) > 2$

$\chi_{\mathcal{B}}(R_\alpha) \le 3$ 

Thm 4.1 (Kechris-Solecki-Todorcevic '99)

$\chi_{\mathcal{B}}(G) \le \Delta + 1$

Pf Base $\{U_n\}$ of (V, τ)

Def $f: V \to 2^{\mathbb{N}}$, $2 = \{0,1\}$,

$x \mapsto (\mathbb{1}_{U_1}(x), \mathbb{1}_{U_2}(x), \dots) \in 2^{\mathbb{N}}$

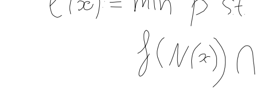
Injective

For $p \in 2^{<\infty}$, def $W_p =$
$= \{ p^\frown q : q \in 2^{\mathbb{N}} \} \subseteq 2^{\mathbb{N}}$, open

$\{ W_p : p \in 2^{<\infty} \}$ is a base for $2^{\mathbb{N}}$

$\forall\, p = (p_1, \dots, p_k)$ $X_p = f^{-1}(W_p) = \bigcap_{i=1}^{k} \begin{cases} U_i, & p_i = 1 \\ U_i^c, & p_i = 0 \end{cases}$
is Borel, so f is a Borel map

Def $\ell: V \to 2^{<\infty}$ by
$\ell(x) = \min p$ st $f(x) \in W_p$ but
$f(N(x)) \cap W_p = \emptyset$



$\ell(x) = 011$

Claim $\forall\, p \in 2^{<\infty}$ $\ell^{-1}(p) \in \mathcal{B}$

Pf Induction on length of p

$$\ell^{-1}(p) = X_p \setminus \Big(\underbrace{N(X_p)}_{\text{Borel by 3.2}} \cup \Big(\bigcup_{q < p} \ell^{-1}(q) \Big) \Big) \square$$

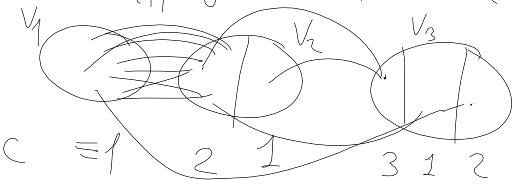
So we have Borel partition

$V = V_1 \cup V_2 \cup V_3 \cup \ldots$ into indep. sets

Construct $c : V_1 \cup \ldots \cup V_i \to [\Delta+1]$

By induction on i:

$x \in V_{i+1}$ get smalles available colour



$c \equiv 1 \quad 2 \quad 1 \quad\quad 3 \quad 1 \quad 2$

$c^{-1}(2) \cap V_2 = \underbrace{N(V_1)}_{\text{Borel by 3.2}} \cap V_2$

$c^{-1}(1) \cap V_2 = V_2 \setminus N(V_1)$, etc $\square$

<u>Def</u> $X \subseteq V$ is <u>r-sparse</u> if

$\forall x \neq y$ in X $\mathrm{dist}_G(x,y) > r$

<u>Lem 4.2</u> (KST'99) Fix $r \in \mathbb{N}$

(a) $\exists$ Borel $c : V \to [k]$ s.t. $\forall i$

$c^{-1}(i)$ is r-sparse, where

$$k = 1 + \sum_{i=1}^{r} \Delta(\Delta-1)^{i-1}$$

(b) $\exists$ Borel r-sparse $X \subseteq V$ s.t. $\forall y \in V$

$\mathrm{dist}_G(y, X) \leq r$.

<u>Pf of (a)</u> Same as in 4.1 (except

$\ell(x) = \min p$ s.t. $f(x) \in W_p$ & $\forall y \neq x$

$(\mathrm{dist}(x,y) \leq r \Rightarrow f(y) \notin W_p)$

For (b), $X = c^{-1}(1)$ $\square$

__Def__ __Borel chr index__ $\chi'_{\mathcal{B}}(G) = \min k$ s.t. $\exists$ Borel edge colouring $c : E \to [k]$ (ie. Borel part $E = E_1 \cup \ldots \cup E_k$ into matchings)

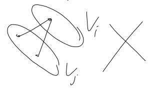
__Thm (KST'99) 4.3__ $\chi'_{\mathcal{B}}(G) \le 2\Delta - 1$

__Pf1__ Apply Thm 4.1 to "line graph" $(E, \{\text{incident edges}\})$ (NB: $(E, \mathcal{B} \restriction E)$ is a SBS by Lem 3.3)

__Pf2__ Apply Lem 4.2 for $r = 2$:
$\exists$ Borel part $V = V_1 \cup \ldots \cup V_k$ into 2-sparse sets.

$$E = \bigcup_{i \ne j} E_{i,j}\,, \quad E_{i,j} = E \cap \big((V_i \times V_j) \cup (V_j \times V_i)\big)$$

$E_{i,j}$ ← a matching



Apply "parallel" greedy over (i,j)) □

__Thm 4.4__ $\exists$ finite S ($|S| \le 2\Delta - 1$) of Borel involutions $V \to V$ ($\forall \varphi \in S$ $\varphi \circ \varphi = Id_X$) st. $G = \text{graph}(S) = (V, \{\{x, \varphi.x\} : x \in V, \varphi \in S, x \ne \varphi x\})$

__Pf__ Thm 4.3 $\Rightarrow$ $\exists$ Borel part $E = \bigcup_{i=1}^{k} E_i$, $k = 2\Delta - 1$

Def $\varphi_i : V \to V$ by $\varphi_i(x) = \begin{cases} y, & xy \in E_i \\ x, & \text{o/w} \end{cases}$

φ_i is Borel by

__Lem 4.5__ SBS X, Y, $f : X \to Y$.
Then f is Borel (measurable) iff $F = \{(x, f(x)) : x \in X\} \in \mathcal{B}(X \times Y)$.

__Pf of "$\Rightarrow$"__ $F^c = \bigcup_k \left(f^{-1}(U_k) \times U_k^c\right)$ for some base $\{U_k\}$ of Y

"$\Leftarrow$": use Lusin thm. □

$S = \{\varphi_1, \ldots, \varphi_k\}$. □

Many consequences:

__Claim__ $\forall$ Borel $A \subset V$ $N(A) \in \mathcal{B}$

__New Pf__: Thm 4.4 $\Rightarrow S$

$$N(A) = \bigcup_{\varphi \in S} \left(\{y \in V : \varphi y \ne y\} \cap \varphi.A\right)$$ □

$G^{\le r} = (V, \{xy : 1 \le \text{dist}(x,y) \le r\})$ is a Borel graph: if $G = \text{graph}(S)$ then $G^{\le r} = \text{graph}(\underbrace{S \times \ldots \times S}_{r})$ □

Thm 4.6 (Marks '16): $\forall d \geq 3$

5 Borel determinacy

A ctble set

$W \subseteq A^{\mathbb{N}}$

Game $G(W)$

I	a_1		a_3		a_5		$a_i \in A$
II		a_2		a_4		$\dots$	

I wins if $(a_1, a_2, \dots) \in W$

Strategy for I: $S : \bigcup_{\text{even } k \geq 0} A^k \longrightarrow A$

— " — II — " — odd k — " —

A Run of S is any $(a_1, a_2, a_3, \dots) \in A^{\mathbb{N}}$ s.t. $a_1 = S(\emptyset)$ & $\forall$ even k $a_k = S(a_1, \dots, a_{k-1})$

S is winning for I if $\forall$ run belongs to W

$G(W)$ is determined if I or II has a winning strategy

Lem 5.1 (AC) $\exists$ non-determined game for $A = \{0,1\}$

Pf Well-order all ($2^{\aleph_0}$) strategies st $\forall S$ $|\{S' : S' < S\}| < 2^{\aleph_0}$

Def W:

given S pick $\alpha_S \in \underbrace{\text{run of } S}_{2^{\aleph_0}} \setminus \underbrace{\{\alpha_{S'} : S' < S\}}_{< 2^{\aleph_0}}$ & if S is II's strategy, add to W $\square$