Descriptive Combinatorics

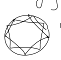
1 Introduction

Ex 1.1 Fix $\alpha \in \mathbb{R} \setminus \mathbb{Q}$

$V = [0,1)$

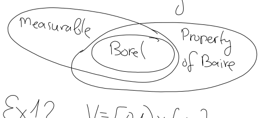
$T: V \to V, \quad x \mapsto x+\alpha \pmod 1$

$E = \{ \{x,y\} \in V^2 : Tx = y \}$

$G = (V,E) = R_\alpha$  $\theta = 2\pi\alpha$

Chromatic number $\chi(R_\alpha) = 2$, assuming AC (Axiom of Choice)

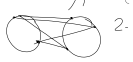
Constrive colouring?



Ex 1.2 $V = [0,1) \times \{0,1\}$

$T(x,i) = (x+\alpha \bmod 1, 1-i)$

$D_\alpha = (V,E)$, $E = \{\{v, Tv\} : v \in V\}$

 2-regular, acyclic

2 Polish Spaces

Topological space (X,τ) [or X]

$\tau = \{\text{open sets}\} \subseteq 2^X$

X is Polish:

- comletely metrizable (ie $\exists$ metric d on X, giving τ st d is complete)
- separable ($\exists$ ctbl dense set D)

$\Rightarrow \exists$ coutable base $\{U_n\}_{n \in \mathbb{N}}$ of τ

Examples:
- $(X, 2^X)$ for ctbl X ✓
- $\mathbb{R}$: $d(x,y) = |x-y|$
- $(0,1]$: $d(x,y) = \left|\frac{1}{x} - \frac{1}{y}\right|$

Prop 2.1 If $X, X_1, X_2, \ldots$ are Polish then the following are also Polish:

(a) $\forall$ closed $Y \subseteq X$

(b) $\forall$ open $Y \subseteq X$

(c) $\bigsqcup_n X_n$

(d) $\prod_n X_n$

(e) $\forall$ G_δ-subset $Y \subseteq X$ (ie $\exists$ open $W_1, W_2, \ldots$ st $Y = \bigcap_n W_n$)

Pf

(b) $d_Y(x,y) = d(x,y) + \left| \frac{1}{\underset{x \in Y}{d(x, Y^c)}} - \frac{1}{d(y, Y^c)} \right|$

(c) Pick d_n on X_n

$d'_n(x,y) = \min\{1, d_n(x,y)\}$

$d_{\bigsqcup_n X_n}(x,y) = \begin{cases} d'_n(x,y) & \text{if } x,y \in X_n \\ 1, & \end{cases}$

(d) $d_{\prod_n X_n}((x_i),(y_i)) = \sum_n \frac{1}{2^n} d'_n(x_n, y_n)$

(e) $Y \cong \{(u, u, \ldots) \in \underbrace{\prod_n W_n}_{\text{closed in } \prod_n}\}$

$\square$

NOT Polish: $\mathbb{Q} \subseteq \mathbb{R}$

3 Borel sets & graphs

<u>σ-algebra</u> on X: susbet of 2^X, $\ni \emptyset$, closed under complements & countable unions

For $\mathcal{F} \subseteq 2^X$, $\sigma(\mathcal{F})$ = smallest σ-alg on X containg $\mathcal{F}$

Borel σ-algebra of a Polish X is $\mathcal{B}(X)$ (or $\mathcal{B}$) is $\sigma(\tau)$

$\Sigma^0_1 = \tau \subsetneq \Sigma^0_2 = \{F_\sigma\text{-sets}\} \subsetneq \Sigma^0_3 \dots$

$\Pi^0_1 = \{\text{closed sets}\} \subsetneq \Pi^0_2 = \{G_\delta\text{-sets}\} \subsetneq \Pi^0_3 \dots$

$\Pi^0_\eta = \neg\Sigma^0_\eta = \{X\setminus Y : Y \in \Sigma^0_\eta\}$

$\Sigma^0_\eta = \{\bigcup_{n=1}^{\infty} A_n : A_n \in \Pi^0_{\eta_n} \text{ with } \eta_n < \eta\}$

for $\eta < \omega_1$ (1st uncountable ordinal)

Then $\mathcal{B} = \bigcup_{\eta<\omega_1} \Sigma^0_\eta$

$1, 2, 3, \dots \omega, \quad \omega+1 \quad \omega+2 \quad \dots \omega+\omega$

Borel hierarchy