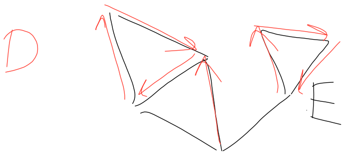
__Thm 5.3__ If $\forall$ comp of G has a cycle, then $\exists$ Borel 1-out-regular digraph $D \subseteq E$



__Thm 4.4__ $\Rightarrow$ $\exists$ finite set $S = \{\varphi_1, \dots, \varphi_k\}$ of Borel involutions $V \to V$ st

- $E = \mathrm{graph}(S) = \Big\{ \{x, \varphi x\} : x \in V, \varphi \in S \Big\} \setminus \mathrm{Diag}$
- $\forall\, xy \in E \ \exists!\, \varphi \in S \ \ \varphi x = y$

__Lem 5.4__ $\forall$ Borel $\varphi : V \to V$,
$\mathrm{Fix}(\varphi) = \{x \in V : \varphi(x) = x\}$ is Borel

__Pf__ $\Big\{(x,x) : \varphi(x) = x\Big\} = \Big\{(x, \varphi(x)) : x \in V\Big\} \cap$
$\cap \Big\{(x,x) : x \in V\Big\}$ Borel

__Lusin__ $\Rightarrow$ $\mathrm{Fix}(\varphi) = \underset{\text{cts, 1-to-1}}{\mathrm{Pr}_1}\Big(\{(x,x) : \varphi(x) = x\}\Big)$
is Borel $\square$

Lem 5.5 $\exists$ linea Borel $<$ on V [ie $\{(x,y)\in V^2 : x<y\}$ is $\mathcal{B}(V^2)$]

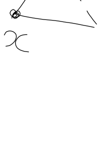
Pf

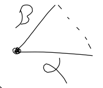
Proof of 4.1 $\Rightarrow \exists$ Borel injective $f: V \to 2^{\mathbb{N}}$

Def $x<y$ if $f(x) <_{Lex} f(y)$ $\square$

Claims deg: $x \mapsto \deg_G(x)$ is a Borel fn

Pf $\deg = \sum_{\varphi\in S} \mathbb{1}_{Fix(\varphi)^c}$ $\square$

Claim $p: x \mapsto \#$  is Borel

Pf $p = \sum_{\alpha<\beta} \mathbb{1}$ 

$= Fix(\alpha)^c \cap Fix(\beta)^c \setminus \bigcup_{\gamma\in S\setminus\{\alpha,\beta\}} Fix(\beta\gamma\alpha)$ $\square$

Lem 5.6 $\forall$ "local" fn is Borel $\square$

$[x]$ (or $[x]_E$) is $\{y \in V : \mathrm{dist}_G(x,y) < \infty\}$

$C = C(E) = \{(x,y) : 0 < \mathrm{dist}(x,y) < \infty\}$
[trans. closure of $E \setminus \mathrm{Diag}$]

(V, C) is $\sqcup$ cliques on comp. of G

Thm 5.7 (Feldman–Moore '77)
$\exists$ Borel labelling $\ell : C \to \mathbb{N}$
s.t. $\forall i \in \mathbb{N}$ $\quad \ell^{-1}(i)$ is a matching.

Pf Construct ℓ:

- label E using $\leq 2\Delta - 1$ labels
- label $\{(x,y) : \mathrm{dist}(x,y) = 2\}$ using some new $4\Delta^2$ labels
- etc

$\square$

<u>aim</u>: Define in a Borel way

$\mathcal{C} = \{\text{some cycles}\}$ s.t.

- $C' \neq C''$ in $\mathcal{C} \Rightarrow V(C') \cap V(C'') = \emptyset$ (vtx disjoint)
- $\forall x \ \exists C \in \mathcal{C} \ \ V(C) \subseteq [x]$

girth: $x \mapsto$ min cycle length in $G \restriction [x]$

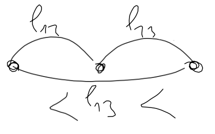
Borel b/c

$$\mathrm{girth}(x) = \inf_{R \in \mathbb{N}} \mathrm{girth}(G \restriction N^{\leq R}(x))$$

is the inf of ctbly many Borel fns.

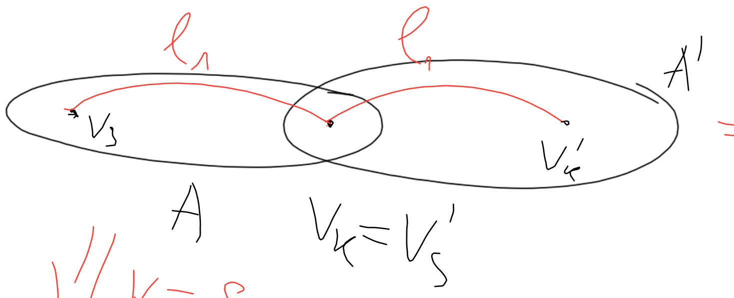
Type: $x \longmapsto$ lex smallest isomorphism type of $(A, <, \ell)$ where $A \subset [x]$ spans a cycle of length $=$ girth(x)

<u>Ex</u> girth $= 3$



$T \rightsquigarrow (\ell_{12}, \ell_{13}, \ell_{23})$

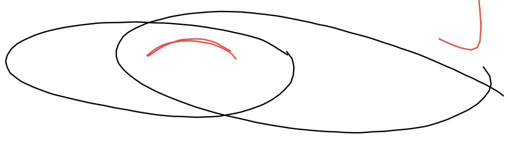
__Fact__ A, A' of the same type $\Rightarrow A \cap A' = \emptyset$

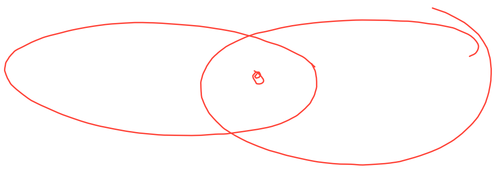


$k \neq s$ $\Rightarrow$ e_1-edges are a matching

✓ $k = s$

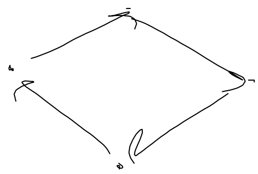
So



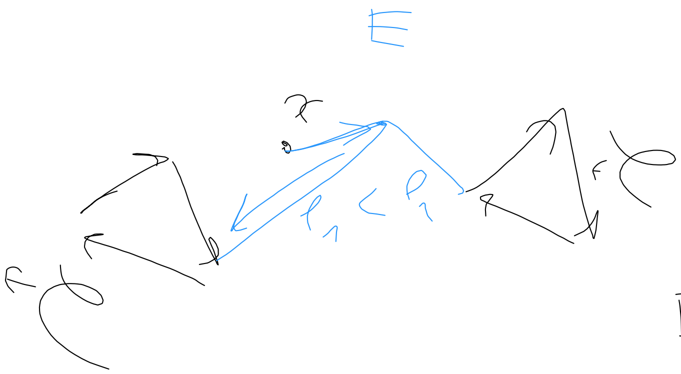


Finishing the proof:

- $\mathcal{C} = \{$cycles of min. type in each comp $[x]\}$
- Orient each cycle in $\mathcal{C}$



- $\forall$ remaining x (ie not on a cycle for $\mathcal{C}$) picks (i) shortest; (ii) lex-min (wrt ℓ) path to a cycle in $\mathcal{C}$



$\square$

6 Baire measurability

Def $A \subseteq X$ is nowhere dense (nd) if $\forall$ open $U \neq \emptyset$ $\exists$ open non-empty $W \subseteq U \setminus A$. [$\Leftrightarrow$ the closure $\bar{A}$ has empty interior]

Ex Cantor set $C \subseteq [0,1]$ is nd

Ex $\mathbb{Q} \subseteq \mathbb{R}$ is NOT nd

Ex $\{x\}$ is nd iff $\{x\}$ is not an open set.

Ex A is nd $\Leftrightarrow$ $\bar{A}$ is nd.

Ex $\forall$ open U $\bar{U} \setminus U$ is nd

Def $A \subseteq X$ is meager if $\exists$ nd $A_1, A_2, \dots$ st $A = \bigcup_{i=1}^{\infty} A_i$

Ex $\mathbb{Q} \subseteq \mathbb{R}$ is meager

Def X is Baire if no open $U \neq \emptyset$ is meager.

Thm 6.1 (Baire Category Theorem) (BCT) $\forall$ complete metric (X,d) is Baire

Pf Take any open $U_0 \neq 0$ & nd $A_1, A_2, \dots$

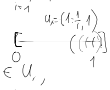
aim $U_0 \not\subseteq \bigcup_{i=1}^{\infty} A_i$

For $i = 1, 2, \dots$ take open $U_i \neq \emptyset$ st.

- $\overline{U_i} \subseteq U_{i-1}$
- $U_i \cap A_i = \emptyset$
- diam $U_i \leq \frac{1}{i}$

Then $W = \bigcap_{i=1}^{\infty} \overline{U_i} = \bigcap_{i=1}^{\infty} U_i$ is non-empty.

$u_i = (1 - \frac{1}{i}, 1)$



Indeed, $\forall i$ pick $x_i \in U_i$,

diam $U_i \to 0 \Rightarrow (x_i)$ is Cauchy

d complete $\Rightarrow \exists x \in X$ $x_i \to x$

$\forall i$ $\overline{U_i} \supseteq \{x_i, x_{i+1}, \dots\}$ contains x

So $x \in W$, i.e. $U_0 \setminus \bigcup_{i=1}^{\infty} A_i \ni x$ // is non-empty. $\square$

Def $A \subseteq X$ is Baire measurable (BM) if $\exists$ open U st. $A \triangle U$ is meager.

$BM(X) = \{BM \text{ sets } A \subseteq X\}$

Prop 6.2 $BM(X)$ is σ-algebra.

Pf
- $\emptyset \in BM(X)$
- $A \in BM$, say $A \triangle U$ is meagre, for some open U, then

$$A^c \triangle \underbrace{(\bar{U})^c}_{\text{open}} \subseteq \underbrace{(A \triangle U)}_{\text{meager}} \cup \underbrace{(\bar{U} \setminus U)}_{\text{nd}}$$

Let $A_1, A_2, \dots \in BM$

(say $A_i \triangle \underbrace{U_i}_{\text{open}}$ is meagre)

$$\left(\bigcup_i A_i\right) \triangle \underbrace{\left(\bigcup_i U_i\right)}_{\text{open}} \subseteq \bigcup_{i=1}^{\infty} (A_i \triangle U_i)$$

ctbl union of meager sets, so meager. $\square$

Corollary 6.3 $\mathcal{B}(X) \subseteq BM(X)$
for Polish X

Pf $\mathcal{B}(X) = \sigma(\{\text{open}\})$
$\underbrace{\subseteq}_{\sigma\text{-algebra}} BM$ as $\forall$ open $U \in BM$ $\square$

Prop 6.4 (Baire alternative)
$\forall A \in BM(X)$ A is meager
or $\exists$ non-empty open U s.t.
$U \setminus A$ is meager
if X is Baire, then ≤ 1 occurs.

Pf

By BM, take open U
s.t. $U \Delta A$ is meagre
$U = \emptyset \Rightarrow A$ meager
o/w $\Rightarrow U \setminus A$ is meager $\square$