<u>Graphing</u> $\mathcal{G}=(V,\mathcal{B},E,\mu)$ s.t.

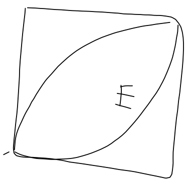
- $(V,\mathcal{B},E)$ is a Borel graph
- μ is a prob. measure on $(V,\mathcal{B})$
- $\exists$ m.p. Borel bijections $\varphi_1,\dots,\varphi_k : V\to V$ s.t. $E=\text{graph}(\varphi_1,\dots,\varphi_k)$
  $\Gamma=\langle \varphi_1,\dots,\varphi_k\rangle$

<u>Ex 8.5</u> Finite $G=(V,E)$ gives

$\mathcal{G}=(V,2^V,E,\text{uniform }\mu)$, a graphing

<u>Ex 8.6</u> $V=(0,1)$, $T: x\mapsto x^2$,

$\mu=$ Leb. measure, $E=\text{graph}(T)$

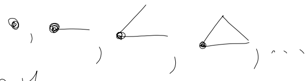


- 2-reg. Borel graph
- not a graphing

<u>Limits of bounded degree graphs</u>

$\mathcal{F}^1 = \{$conn. rooted graphs$\}$

$\mathcal{F}^1_R = \{$rooted graphs of radius $\le R\} \subseteq \mathcal{F}^1$

E.g. $\mathcal{F}^1_1 = \{$  $, \dots \}$

distance on $\mathcal{F}^1$: $d(H,F) :=$
$= 2^{-\min\{r\in\mathbb{N}:\ \mathrm{Ball}_r(H)\not\cong \mathrm{Ball}_r(F)\}}$

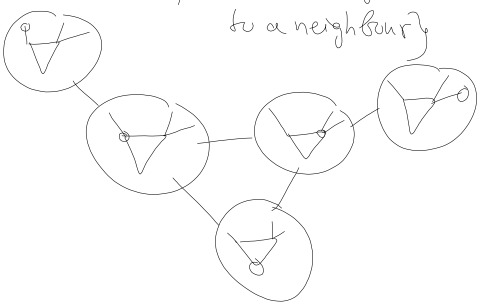
Polish spare

(G_n) is <u>BS-convergent</u> if $\forall r\in\mathbb{N}$
$\mathrm{Ball}_r(G_n, \text{random } v)$ converge as $n\to\infty$

Such (G_n) gives a prob measure ν on $\mathcal{F}^1$ st $\forall F\in\mathcal{F}^1_r$

$\mathbb{P}_{H\sim\nu}(\mathrm{Ball}_r(H)\cong F) = \lim_{n\to\infty} \mathbb{P}(\mathrm{Ball}_r(G_n, \text{rnd } v)\cong F)$

<u>Idea 1</u>: $V=\mathcal{F}^1$, $E=\{HF:$ shift root to a neighbour$\}$



<u>Idea 2</u>: $V=\{(H,\omega): H\in\mathcal{F}^1, \omega: V(H)\to[0,1]\}$

measure μ: sample $H\sim\nu$, then indep. uniform $\omega(v)\in[0,1]$, $v\in V(H)$

$E=\{$shifts of the root$\}$, $\mathcal{G}=(V,\mathcal{B},E,\mu)$

<u>Then</u> $\mathbb{P}_{v\sim\mu}(\mathrm{Ball}_r(\mathcal{G},v)\cong F) = \lim_{n\to\infty}\mathbb{P}(\mathrm{Ball}_r(G_n,\text{rnd } v)\cong F)$

<u>Open</u>:

<u>Aldous-Lyons Conj</u>: Is every graphing a limit of finite G_n?

Canonical form for $\mathcal{G}$?

9 Measurable chromatic numbers

Def $A \subseteq V$ is (Lebesgue) measurable if $\exists$ Borel $B, N \subseteq V$ s.t $B \triangle A \subseteq N$ & $\mu(N)=0$ [$\Leftrightarrow$ $A \in \mathcal{B}_\mu$, completion of $\mathcal{B}$ wrt μ] & $\mu(A)=\mu(B)$

Def Measurable chromatic number

$\chi_\mu(\mathcal{G})$ is min k s.t. $\exists$ partition $V = A_1 \cup \dots \cup A_k$, each $A_i \in \mathcal{B}_\mu$ & is indep

Equivalently, $\chi_\mu(\mathcal{G})$ is min $k \geq \chi(\mathcal{G})$ s.t. $\exists$ Borel X s.t. $\mu(X)=0$ & $\chi_\mathcal{B}(\mathcal{G}-X) \leq k$

Lem 9.1 $\forall$ Borel $A \subseteq V$

$(\mu(A)=0 \Rightarrow \mu([A])=0)$

Pf $[A] = \bigcup_{\gamma \in \Gamma} \underbrace{\gamma A}_{\text{Borel, measure } 0}$, ctbl union

so $\mu([A])=0$

$\square$

Def Approximate chr number

$\chi_\mu^{ap}(\mathcal{G})$ is min $k \geq \chi(\mathcal{G})$ s.t. $\forall \varepsilon > 0$ $\exists X \in \mathcal{B}$ s.t. $\mu(X) < \varepsilon$ & $\chi_\mathcal{B}(\mathcal{G}-X) \leq k$

Trivially $\chi \leq \chi_\mu^{ap} \leq \chi_\mu \leq \chi_\mathcal{B}$

Def φ is ergodic if $\forall$ invariant $A \in \mathcal{B}_\mu$ $\mu(A) \in \{0,1\}$

Lem 92 For $\alpha \notin \mathbb{Q}$, R_α w. Leb measure μ is ergodic

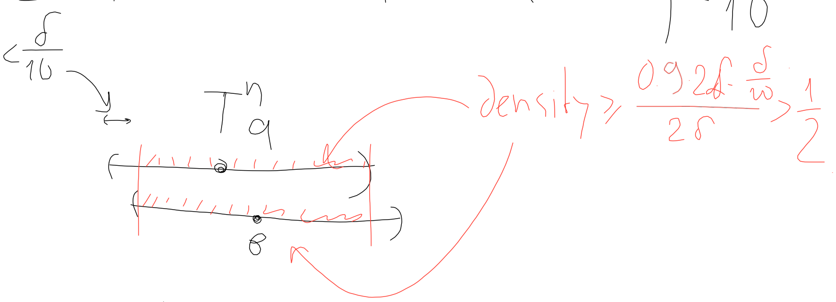
Pf Sse not: inv $A \in \mathcal{B}_\mu$ s.t. A & $B = A^c$ have $\mu > 0$

Leb Density thm: $\exists a\ \exists \varepsilon > 0\ \forall \delta \in (0,\varepsilon)$ $\mu(A \cap (a-\delta, a+\delta)) \geq 0.9 \cdot 2\delta$

$\exists b$ —/— $\mu(B \cap (b-\delta, b+\delta)) \geq \ldots$

Fix a, b & common δ

$\exists\, n,m \in \mathbb{Z}$ s.t. $|a + n\alpha - b + m| < \frac{\delta}{10}$



$\Rightarrow\Leftarrow\ \underbrace{(T^n A) \cap B}_{=\,\emptyset} = A \cap B = \emptyset$

$\square$

Lem 9.3 $\chi_\mu(R_\alpha)=3$, $\chi_\mu^{ap}(R_\alpha)=2$

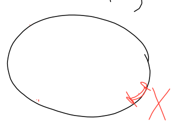
Pf If $[0,1)=A\cup B$, meas. indep

$\Rightarrow TA=B \Rightarrow \mu(A)=\mu(B)=1/2$

$\Rightarrow T^3A=A$

This A contradicts Lem 9.2

$\chi_\mu^{ap}\le 2$:



$\mathcal{G}-X$ has fin. comp, so $\chi_B\le 2$ $\square$

Conley–Kechris '13 $F_n \curvearrowright [0,1]^{F_n}$, shift

$\Rightarrow$ $2n$-regular, acyclic (mod measure 0)

$\chi_\mu^{ap}\ge\left(\frac{1}{2}+o(1)\right)\sqrt{n}$ (while $\chi(\mathcal{G}$-measure 0$)\le 2$)

Conley '17 $\exists\, \mathcal{G}$ $\chi_\mu^{ap}\le\chi_\mu-2$?

Is f s.t. $\forall\, \mathcal{G}$ $\chi_\mu\le f(\chi_\mu^{ap})$?

Thm 9.4 (Conley–Marks–Tucker-Drob '16)

No K_{d+1}, $d\ge 3$, $\Delta(\mathcal{G})\le d\Rightarrow\chi_\mu(\mathcal{G})\le d$

$\Rightarrow \chi_{BM}\le d$

10 Matchings

Ergodicity $T_\alpha \Rightarrow \forall$ Borel matching M in R_α $\mu(V(M)^c)>0$

Laczkovicz'88 Same conclusion with $\chi_\mathcal{B}(\mathcal{G})=2$ (2-regular)

Conley-Kechris'13 2d-regular, $d\geq 2$

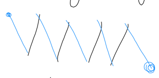
Open d-regular bip, w. odd $d\geq 3$?

Thm 10.1 (Elek-Lippner'10)

$\forall$ Borel graph $G=(V,\mathcal{B},E)$

$\exists$ Borel matchings $M_0=\emptyset, M_1, M_2, \ldots$ st

(a) $\forall i\geq 1$ M_i has no augmenting path of length $\leq 2i-1$



(b) $\forall i\geq 0$ M_{i+1} is obtained from M_i by augmenting paths of lengths $\leq 2i+1$

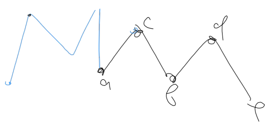
Pf Given M_i, let $N_0=M_i$.

Lem 4.2 gives $(2i+1)$-sparse Borel $c: V\to$ finite set S

Let $(c_j)_{j=1}^{\infty}$ be seq of seq s.t. $\forall w\in S^{\leq 2i+1}$ appears inf. often

Inductively on j

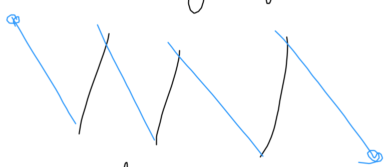
let N_{j+1} be obtained from N_j by augmenting (in parallel) coloured as c_j



(vertex disjoint)

$$M_{i+1} = \bigcup_{m} \bigcap_{j \geq m} N_j$$

- Borel matching
- no aug. path of length $\leq 2i+1$



M_{i+1}

$\square$

<u>HW Lem</u>: M in fin. graph G

s.t. no aug. path of length $\leq 2n+1$

Then $|M| \geq |M_{max}| - \frac{2}{n} |V|$