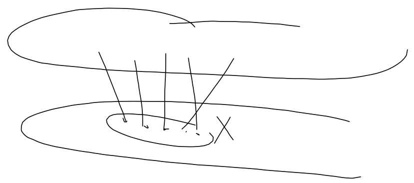
Thm 7.4 (Rado): Bipartite locally finite H has a PM iff

(Hall) $\forall$ finite X in a part
$$|N(X)| \geq |X|.$$

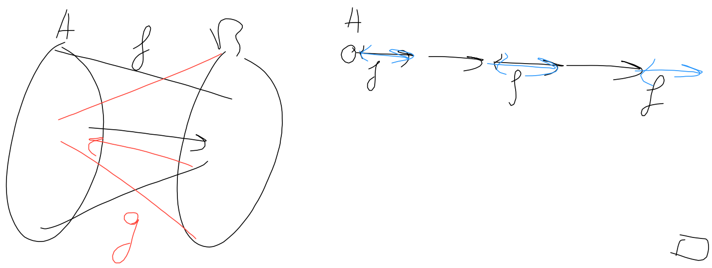
Pf $\Leftarrow$ (Finite) Hall Thm $\Rightarrow$

$\forall$ finite X in a part can be matched



AC $\Rightarrow$ each part can be matched into other

Use Banach-Bernstein-Schroeder Lemma



$\square$

Def Bip H has **$\mathrm{Hall}_{\varepsilon,n}$** if

- (Hall) holds
- $\forall$ finite X in a part, connected in $H^{\leq 2}$, $|X| \geq n \Rightarrow |N(X)| \geq (1+\varepsilon)|X|$

Pf of 7.3 G has $\mathrm{Hall}_{\varepsilon,1}$

$M_0 = \emptyset$, $\varepsilon_0 = \varepsilon$, $f(0) = 1$

Take $\varepsilon_0 > \varepsilon_1 > \varepsilon_2 > \dots$ (>0)
& increasing $f(n)$ $\left[\frac{8}{f(n)} \leq \varepsilon_n - \varepsilon_{n-1} \right]$

Lem 7.2 $\Rightarrow$ Borel $A_0 \cup A_1 \cup \dots = V$
s.t. $A_0 = [A_0]$ is meager
& $\forall n \geq 1$ A_n is $f(n)$-sparse

Strategy $\underset{\text{Borel}}{\vee}$ $M_0 \subseteq M_1 \subseteq M_2 \subseteq \dots$ in G
s.t. I_n
- $V(M_n) \supseteq A_1 \cup \dots \cup A_n$
- $G - M_n$ has $\mathrm{Hall}_{\varepsilon_n, f(n)}$

I_0 ✓

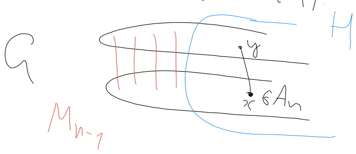
$I_{n-1} \Rightarrow I_n$:

$H = G - M_{n-1}$ has I_{n-1}

aim $M_n = M_{n-1} \cup M_n'$ s.t. I_n holds

$U = (V(M_{n-1}))^c$, unmatched

claim $\forall x \in U \cap A_n\ \exists xy \in E(H)$
s.t. $H - e$ has (Hall)



(follows from Rado Thm (7.4))

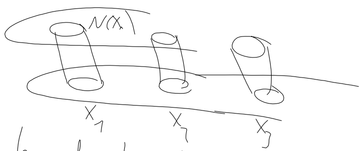
Let $M_n' = \{xy : x \in U \cap A_n$ &
$y \in N(x)$ is min. suitable$\}$

let $M_n = M_{n-1} \cup M_n'$

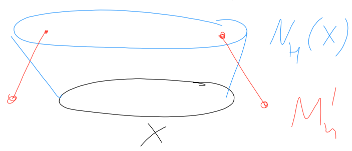
Check I_n:

$V(M_n) \supseteq A_n \cap U$ ✓

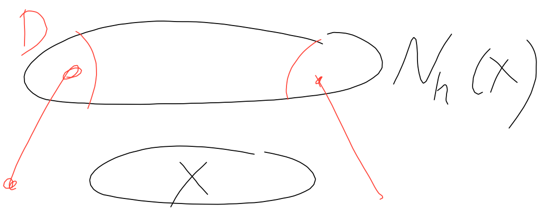
let X be finite set in a part
s.t. X is $H^{\leq 2}$-connected



Need to check Hall & $\mathrm{Hall}_{\varepsilon_n, f(n)}$
for this X



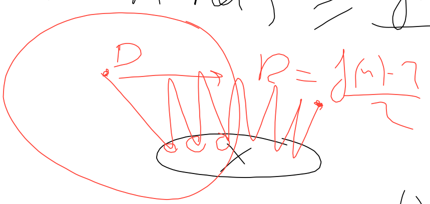
$D = N_H(X) \cap V(M_n')$
$= N_H(X) \setminus N_{H'}(X)$, $H' = H - M_n'$



<u>Case 1</u> $|D| \geq 2$

Then D is $(f(n)-2)$-sparse

So $N^{\leq \frac{f(n)-2}{2}}(y) \cap X$, $y \in D$, are disjoint, & (since $|D| \geq 2$) each has $\geq \frac{f(n)-4}{4}$ vertices



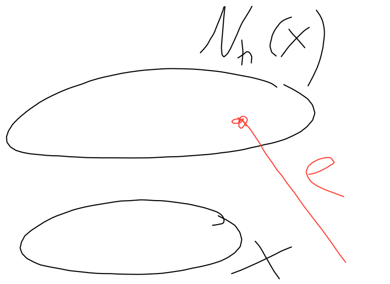
So $|D| \leq \frac{|X| \cdot 4}{f(n)-4} \leq \frac{|X| \cdot 8}{f(n)}$

& $|N_{h'}(X)| = |N_h(X)| - |D|$

$$\geq (1+\varepsilon_{n-1})|X| - \frac{|X| \cdot 8}{f(n)}$$

$$\geq (1+\varepsilon_n)|X|$$

Case 2 $|D| \le 1$



(Hall): if $D = \emptyset$, then use Hall(H)

$|N_{H'}(X)| = |N_H(X)| \ge |X|$ ✓

$|D| = 1 \Rightarrow e$ is a PM of H, with $X \subseteq N_H(X) \setminus D$

so $|N_{H'}(X)| \ge |X|$

if $|X| \ge f(n)$, then

$|N_{H'}(X)| \ge |N_H(X)| - 1 \ge (1+\varepsilon_{n-1})|X| - 1$

$\ge (1+\varepsilon_n)|X|$

$1 \le \frac{|X|}{f(n)}$

$\square$

Borel Marczewski Problem:
Dis/prove: $\mu : \{\text{Bounded Borel sets in } \mathbb{R}^3\} \to [0,\infty)$
s.t. • $\forall$ disjoint A, B, $\mu(A \cup B) = \mu(A) + \mu(B)$ & • $\forall A$ & isometry γ
$\mu(\gamma(A)) = \mu(A)$ • $\mu([0,1]^3) = 1$
$\overset{?}{\Longrightarrow}$ $\mu(A)$ = Lebesgue measure of A?

8. Graphings

Def A *graphing* $\mathcal{G} = (V, \mathcal{B}, E, \mu)$ where $(V, \mathcal{B}, E)$ is a Borel graph & μ is a probability measure on $(V, \mathcal{B})$ s.t. $\exists$ Borel bijections $\varphi_1, \dots, \varphi_k : V \to V$ s.t.

(a) $E = \text{graph}(\varphi_1, \dots, \varphi_k) = \{\{x,y\} : x \in V,\ y = \varphi_i(x) \text{ some } i\}$

(B) each φ_i is *measure-preserving*

(i.e. $\forall A \in \mathcal{B}$ $\varphi_i^{-1}(A)$ has the same measure as A, i.e. $\mu(A) = \mu(\varphi_i^{-1}(A))$)

Lem 8.1 $\mathcal{G}$ is a graphing $\overset{(m.p.)}{\&}$ $A, B \in \mathcal{B}(V)$, $f : A \to B$ Borel bijection, s.t. $\forall x \in A$ $f(x) \in [x]_E$
Then f is m.p.

Pf $\Gamma = \langle \text{"}\varphi_1\text{"}, \dots \text{"}\varphi_k\text{"} \mid \rangle$
Enumerate it $\Gamma = \{\gamma_0, \gamma_1, \dots\}$

$|\gamma_0| \le |\gamma_1| \le |\gamma_2| \le \dots$
"$\emptyset$" "φ_1" "φ_2"

For $i \geq 0$ define, $D_{\gamma_i} = \{x \in A : f(x) = \gamma_i x\} \setminus (D_0 \cup \dots \cup D_{i-1})$

$D_\emptyset = \{x : f(x) = x\}$
$D_{\varphi_1} = \{x \in A : f(x) = \varphi_1(x)\} \setminus D_\emptyset$
$\vdots$

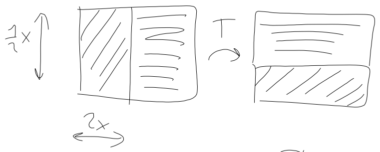
Each D_{γ_i} is Borel
& if $\gamma_i = \varphi_{k_1}^{\varepsilon_1} \circ \dots \circ \varphi_{k_n}^{\varepsilon_n}$
then $f|_{D_{\gamma_i}}$ is $\varphi_{k_1}^{\varepsilon_1} \circ \dots \circ \varphi_{k_n}^{\varepsilon_n}|_{D_{\gamma_i}}$ $\lfloor$ m.p.
& D_{γ_i} are disjoint, & partition A
So f is m.p. $\square$

Then μ is also called *invariant* (or *unimodular*)
(see also Mass Transport Principle)

Example 8.1 R_α 
$= \text{graph}(x \mapsto x + \alpha \pmod 1)$
$V = [0,1)$
& μ = Lebesgue measure λ

Ex 8.3 Borel m.p. actions $\Gamma = \langle s_1, \dots, s_n \rangle$ on a prob space $(V, \mathcal{B}, \mu)$

Ex 84 "Dough cutting"



$[0,1]^2 \cong 2^{\mathbb{Z}}$ (except ctbl set)

$(0 b_1 b_2 b_3 \dots, 0 c_1 c_2 c_3 \dots) \Rightarrow (\dots c_3 c_2 c_1 \underset{\uparrow}{b_1} b_2 b_3 \dots)$

$T(\dots c_2 c_1 \underset{\substack{\uparrow \\ 0}}{b_1} b_2 b_3) = (\dots c_2 c_1 b_1 \underset{\substack{\uparrow \\ 0}}{b_2} b_3 \dots)$

λ on $[0,1]^2 \longleftrightarrow$ product measure on $2^{\mathbb{Z}}$

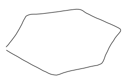
Graphings as limits of bounded degree graphs

Graphs w $\Delta \le d$ ← Benjamini-Schramm

(G_n) is $\underline{\text{BS conv}}$ to a graphing $\mathcal{G}$

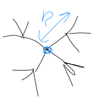
if $\forall R \in \mathbb{N}$, $\forall$ rooted F: $\mathbb{P}_{x \sim \mathrm{Unif}(V(G_n))}\left(N^{\le R}_{G_n}(x) \cong F\right)$

$\longrightarrow \mathbb{P}_{x \sim (V,\mu)}\left(N^{\le R}_{\mathcal{G}}(x) \cong F\right)$

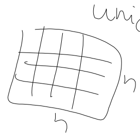
Eg $G_n = C_n$  n-cycle

Then $G_n \xrightarrow{BS} R_\alpha$

$\underline{\text{Eg}}$ G_n is random 4-reg graph on n vertices

Then $N^{\le r}_{G_n}(x) \cong$ 

Possible $\mathcal{G}$: $(S^2, \mathcal{B}, \mathrm{graph}(\varphi_1, \varphi_2)$, uniform measure) "generic"

$\underline{\text{Ex}}$ G_n 

possible

$\mathcal{G} = ([0,1]^2, \mathcal{B}, \mathrm{graph}((x,y) \mapsto (x+\alpha \bmod 1, y), (x,y) \mapsto (x, y+\beta)), \lambda)$, $\alpha, \beta \notin \mathbb{Q}$