A is BM if $A = U \Delta M$ for some open U & meager

<u>BM chromatic number</u> $\chi_{BM}(G)$ is min k s.t. $\exists$ partition $V = V_1 \cup \ldots \cup V_k$ s.t. $\forall i$ V_i is a BM indep set

$\chi \leq \chi_{BM} \leq \chi_B$

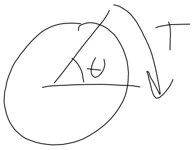
for $x \in V$ $[x] = \{y \in V : \mathrm{dist}(x,y) < \infty\}$

$A \subseteq V$ $[A] = \bigcup_{x \in A} [x]$

$A \subseteq V$ is $(G\text{-})$ <u>invariant</u> if $A = [A]$

G is <u>generically ergodic</u> if $\forall$ invariant BM A, A or A^c is meager

Lem 6.5 $R_\alpha = \text{graph}(T)$, $\alpha \notin \mathbb{Q}$



$\theta = \frac{\alpha}{2\pi}$ is generically ergodic

Pf Let $A \in BM$ be invariant
($\Leftarrow$) $TA = A$)

Write $A = U \Delta M$, $A^c = W \Delta N$,
U, W open, N, M meager
Assume $U, W \neq \emptyset$ (o/w done)
$\exists k$ s.t. $X := T^k U \cap W$ is not $\emptyset$

Then $X = \underset{[X \subseteq W]}{(A \cup X)} \cap (T^k A^c \cap X)$

$\subseteq (A \cap W) \cup T^k(A^c \cap \underbrace{T^{-k} X}_{\subseteq U})$

$\subseteq \underset{\text{meager}}{N \cup T^k M} \Rightarrow\Leftarrow$ BCT (6.1)

$\square$

A is comeager in U $\overset{\text{def}}{\Longleftrightarrow}$ $U \setminus A$ is meager

<u>Cor 6.6</u>: $\chi_{BM}(R_\alpha) \geq 3$.

<u>Pf</u> Suppose $V = A \sqcup B$, indep, BM.

then $TA = B$ & $TB = A$

i.e. $T^2 A = A$

So A is $R_{2\alpha}$-invariant

Lem 6.8 $\Rightarrow$ A or A^c is meager

E.g. A meager $\Rightarrow$ $B = TA$ is meager

$V = A \cup B$ $\Rightarrow\Leftarrow$ BCT. $\square$

7 Banach-Tarski Paradox (BTP)

Def $A, B \subseteq \mathbb{R}^n$ are <u>equidecomposable</u> $(A \sim B)$ if $\exists$ finite partitions $A = A_1 \sqcup \ldots \sqcup A_n$ & $B = B_1 \sqcup \ldots \sqcup B_n$, & isometries $\varphi_1, \ldots, \varphi_n$ of $\mathbb{R}^n$ s.t. $\forall i \ \varphi_i(A_i) = B_i$

<u>BTP (1924)</u> $B \sim B \sqcup B$, $B \subseteq \mathbb{R}^3$ unit Ball

<u>Dougherty & Foreman '92</u>

$B \sim B \sqcup B$ with BM

<u>AC-free version</u>: $\exists$ open disjoint $U_1, \ldots, U_n$ & iso $\varphi_1, \ldots, \varphi_n$ s.t. $\overline{\bigcup U_i} = B$, $\varphi_1 U_1, \ldots, \varphi_n U_n$ are disjoint, & $\overline{\bigcup \varphi_i U_i} = B \sqcup B$

<u>Today</u> new proof by Marks-Unger '16

$A \sim B$, with given $\varphi_1, \ldots, \varphi_n$ iff Bip. graph $(A, B, \{\varphi_1, \ldots, \varphi_n\})$ w. $E := \{(a, b) \in A \times B : \exists i \ \varphi_i a = b\}$ has a perfect matching (PM)

<u>Lem 7.2</u>: $\forall$ Borel gr. G, $\forall$ fn $f : \mathbb{N} \to \mathbb{N}$

$\exists$ Borel partition $V = A_0 \sqcup A_1 \sqcup \ldots$ s.t. A_0 is meager & $\forall i \geq 1$ A_i is <u>$f(i)$-sparse</u> (i.e. $\forall x \neq y$ in A_i $\text{dist}(x, y) > f(i)$) & $[A_0] = A_0$

<u>Pf</u> Base $\{U_i\}$ of open set

$B_{i,r} = \{x \in V : N^{\leq r}(x) \cap U_i = \{x\}\} \in \mathcal{B}$

<u>Claim</u> $B_{i,r}$ is r-sparse

<u>Claim</u> $\forall r \ \bigcup_i B_{i,r} = V$

<u>Idea</u> $\forall n$ let $A_n' = B_{k, f(n)}$ s.t. not meager in U_n (use BCT) then let $A' = \bigcup_n A_n'$, $A_0 = [X \setminus A']$ & $A_n = A_n' \setminus A_0$

<u>Issue</u> A' is comeager but A_0 need not be meager!

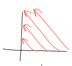
Fix Borel $\varphi_1, \ldots, \varphi_k : V \to V$ that generate G as in thm 4.4

$\Gamma = \langle \varphi_1, \ldots, \varphi_k \rangle$, group

Enumerate $\Gamma = \{T_1, T_2, \ldots\}$

E.g. $y \in [x] \iff \exists T_i$ s.t. $T_i x = y$

Fix bijection $n : \mathbb{N}^2 \to \mathbb{N}$



<u>BCT</u> $\forall n = n(i, j)$ $\exists k$ s.t. $T_i(B_{k, f(n)})$ is not meager in U_j

let $A_n' = B_{k, f(n)}$ for such k

& $A' = \bigcup_n A_n'$, $A_0 = [V \setminus A']$, $A_n = A_n' \setminus A_0$

(NB: $\forall B_{k, f(n)}$ is Borel $\Rightarrow T(B_{k, f(n)}) \in \mathcal{B} \subseteq BM$) $\square$

Thm 7.3 (Marks-Unger '16)

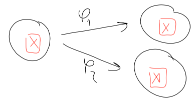
G Bipartite w parts V_1 & V_2, $\exists\, \varepsilon>0$ st. $\forall$ finite $X \subseteq V_1$

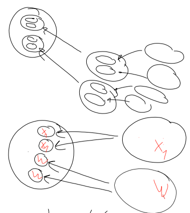
(Hall$_\varepsilon$) $|N(X)| \geq (1+\varepsilon)|X|$

Then $\exists$ Borel matching M st $V(M)^c$ is meager & invariant

Cor BTP w BM pieces

$V_1 = B$, $V_2 = B \sqcup B$





Then Hall($\varepsilon=1$) holds

Apply Thm 7.3 $\square$

$V_1 = B$, $V_2 = B_1 \sqcup B_2$, Borel sets

$\varphi_1, \dots, \varphi_n$: isometries (Borel as a map)

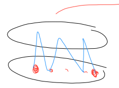
$E = \{ab \in V_1 \times V_2 \mid \exists \varphi_i \; \varphi_i a = b\}$

Pf of 7.3

Def Bip H has Hall$_{\varepsilon,n}$ if $\forall$ finite $X \subseteq$ a part

- (Hall) $|N(X)| \geq |X|$
- if $|X| \geq n$ & X is $H^{\leq 2}$-connected

$\Rightarrow |N(X)| \geq (1+\varepsilon)|X|$



Note: Hall$_{\varepsilon,1} \equiv$ Hall$_\varepsilon$

Let $M_0 = \emptyset$, $\varepsilon_0 = \varepsilon$, $f(0) = 1$

Take $\varepsilon_0 = \varepsilon > \varepsilon_1 > \varepsilon_2 > \dots > 0$

& fast growing $f: \mathbb{N} \to \mathbb{N}$

$\underline{\text{Lm 7.2}} \Rightarrow V = A_0 \cup A_1 \cup \ldots$, Borel,

s.t. A_n is $f(n)$-sparse, $A_0 = [A_0]$ is meager.

<u>Strategy</u>: Borel matchings

$$M_0 \subseteq M_1 \subseteq M_2 \subseteq \ldots$$

s.t. $V(M_n) \supseteq \bigcup_{i=1}^{n} A_i$

<u>Induction assumption I_n</u>:

$G - M_n$ has $\mathrm{Hall}_{\varepsilon_n, f(n)}$

I_0 ✓ $(\mathrm{Hall}_{\underbrace{\varepsilon_0, f(0)}_{\varepsilon, 1}} = \mathrm{Hall}_\varepsilon)$

Sse I_{n-1},

ie $H := G - M_{n-1} \in \mathrm{Hall}_{\varepsilon_{n-1}, f(n-1)}$

<u>aim</u> $M_n = M_{n-1} \cup M_n'$ s.t.

$V(M_n') \supseteq A_n \cap U$, $U = V(H)$

<u>claim</u> $\forall x \in A_n \cap U \;\; \exists y \in N(x) \cap U$

s.t. $H - \{x, y\}$ has Hall (b/c

H has a PM by Rado's thm)

$M_n' := \{xy : x \in A_n \cap U$, min $y \in N(x)$ as in the claim$\}$.