On Envy-Free Cake Division

Oleg Pikhurko

The old problem introduced by Gamov and Stern [3], to divide a cake between m people so that each person thinks that he has at least as much cake as anybody else (*envy-free division*), has been solved by Brams and Taylor [1]. This discovery attracted further interest to the area and, a few years later, Robertson and Webb [4] found a new construction. Here we present another solution, which is similar in spirit to the latter.

Let each player P_i define a measure μ_i on the cake C such that $\mu_i(C) = 1$, $i \in [m] \equiv \{1, \ldots, m\}$. We assume that P_i can always cut off a piece $B \subset A$ with $\mu_i(B) = c$ for any $A \subset C$ and $0 \leq c \leq \mu_i(A)$. We want to find a procedure for constructing a partition $C = W_1 \cup \cdots \cup W_m$ with $\mu_i(W_i) \geq \mu_i(W_j)$ for all $i, j \in [m]$.

We use the following lemma from [4], for which we present an elementary proof.

Lemma 1 *For every $A \subset C$, $n \in \mathbb{N}$ and $\varepsilon > 0$, there is a partition $A = Y_1 \cup \cdots \cup Y_n$ such that, for every $j \in [n]$, we have $\mu_m(Y_j) = \mu_m(A)/n$ (exactly) and*

$$|\mu_i(Y_j) - \mu_i(A)/n| < \varepsilon \quad \textit{for all } i \in [m-1]. \tag{1}$$

In particular, by combining sufficiently small pieces, we can approximate the division of a piece of cake into any desired ratio, to any desired closeness.

Proof. Our proof uses induction on m. Suppose that $m > 1$ and we are given some n and ε. By the induction assumption applied to $P_1, \ldots, P_{m-1}$ with respect to nt and ε' (specified later), find an appropriate partition $A = Y_1 \cup \cdots \cup Y_{nt}$.

We may assume that P_m has ordered $Y_1, \ldots, Y_{nt}$ in decreasing order of measure. Let $X = Y_1 \cup \cdots \cup Y_n$, and $X_i = Y_{n+i} \cup Y_{2n+i} \cup \cdots \cup Y_{(t-1)n+i}$ for each $i \in [n]$.

In μ_m, among the pieces $X_1, \ldots, X_n$, X_1 is the largest and X_n is the smallest. Since

$$\mu_m(X_1) - \mu_m(X_n) = \mu_m(Y_{n+1}) - (\mu_m(Y_{2n}) - \mu_m(Y_{2n+1})) - \cdots - \mu_m(Y_{nt}) \leq \mu_m(Y_{n+1}),$$

there is enough cake in X for P_m to distribute among $X_1, \ldots, X_n$ to form equal pieces $Z_1, \ldots, Z_n$.

Since $X_i \subset Z_i \subset X_i \cup X$ for each $i \in [n]$, we have

$$(t-1)\left(\frac{\mu_j(A)}{nt} - \varepsilon'\right) < \mu_j(Z_i) < (t-1+n)\left(\frac{\mu_j(A)}{nt} + \varepsilon'\right) \quad \text{for all } j \in [m].$$

Thus, given ε and n we can choose t sufficiently large, and then ε' sufficiently small to satisfy $|\mu_j(Z_i) - \mu_j(A)/n| < \varepsilon$. $\square$

At each stage of our algorithm we consider a specific piece of cake A, beginning with $A = C$. The players are divided into groups $G_1, \ldots, G_k$ so that P_i and P_j belong to one group if and only if $\mu_i(A) = \mu_j(A)$. (Thus $k = 1$ initially.)

During the division of A and $C \setminus A$, if all members of each group agree with each other on all of the arising pieces, we show that an evey-free division has been found. If there is a disagreement within any group, then we find a new piece A for which there are more groups. Since there are m players and so at most m groups, this procedure must terminate.

Let us describe the procedure. First assume that the members of each group G_i agree with each other on every step; let us denote this common measure by η_i. Suppose G_i has m_i members; denote

$$a_i = \eta_i(A), \quad b_i = m_i(b - d(1-a_i)^2) \quad \text{and} \quad c_i = m_i(c - da_i^2) \quad \text{for each } i \in [k], \tag{2}$$

where b and c are chosen so that $\sum_{i=1}^k b_i = \sum_{i=1}^k c_i = 1$ and $d > 0$ is small enough to ensure that all b_i's and c_i's are non-negative. (For example, $d = 1/m$ would do as $b, c \geq 1/m$.)

Suppose (hypothetically) that we can find partitions $A = U_1 \cup \cdots \cup U_k$ and $C \setminus A = V_1 \cup \cdots \cup V_k$ that divide A and $C \setminus A$ in the proportions

$$b_1 : b_2 : \cdots : b_k \quad \text{and} \quad c_1 : c_2 : \cdots : c_k, \tag{3}$$

respectively (in *each* measure $\eta_1, \ldots, \eta_k$). Then letting $W_i = U_i \cup V_i$, we have, for every distinct $i, j \in [k]$,

$$\frac{\eta_i(W_i)}{m_i} - \frac{\eta_i(W_j)}{m_j} = \frac{b_i a_i + c_i(1-a_i)}{m_i} - \frac{b_j a_i + c_j(1-a_i)}{m_j} = d(a_i - a_j)^2 > 0. \quad (4)$$

Thus, if G_i receives W_i, for each $i \in [k]$, then each group considers its share (per one member) the largest. It is surprising that this can be achieved by simply splitting A and $C \setminus A$ into certain proportions, namely (3). Once one believes in the existence of such proportions, it is not hard to find them, but the whole affair seems just a bit of good luck.

In reality, by Lemma 1, we can ensure only that the partitions we build are ε-close to (3), that is, satisfy $|\eta_j(U_i) - b_i\eta_j(A)| < \varepsilon$ and $|\eta_j(V_i) - c_i\eta_j(X \setminus A)| < \varepsilon$ for all $i, j \in [k]$. But this is fine, as the left-hand side of (4) is still strictly positive provided ε is sufficiently small. Next, again by Lemma 1, partition $W_i = U_i \cup V_i$ into m_i parts that are equal from the point of view of G_i, while the remaining groups consider them 'sufficiently equal'; these m_i parts are distributed among the members of G_i, $i \in [k]$.

The players cannot envy each other by (4), so to complete the proof we have to describe how to handle disagreements within a group about some piece $U \subset C$. Clearly, at least one of $U \cap A$ or $U \setminus A$ is still 'disputable'. Cutting this piece further (using Lemma 1, for example), we can find a 'disputable' piece V smaller than $\min_{1 \le i < j \le k} |\eta_i(A) - \eta_j(A)| > 0$ in each player's measure. Then replacing A by $A \triangle V$, we obtain more groups, as required.

Although the number of cuts in the algorithm is always finite, we cannot bound it by a function of m only. A pleasant property of our algorithm is that it requires a bounded number of *cake transfers*: if the players live in different cities and send each other pieces of cake by post, then the number of dispatches is $O(m^2)$.

Our construction admits some extensions that we mention without detail. For example, the algorithm solves the *weighted envy-free* problem when we are given positive numbers $\alpha_1, \ldots, \alpha_m$ summing up to 1 and we look for a partition $C = W_1 \cup \cdots \cup W_m$ such that $\mu_i(W_i)/\alpha_i \ge \mu_i(W_j)/\alpha_j$; we let $b_i = (b - d(1-a_i)^2) \sum_{P_j \in G_i} \alpha_j$, etc. Furthermore, making d small, we can additionally ensure that each $\mu_i(W_j)$

is arbitrarily close to α_j. If each player wants to minimise his share, we let $b_i = m_i(b + d(1 - a_i)^2)$, $c_i = m_i(c + da_i^2)$, etc.

Our construction can be also reworded into what Even and Paz [2] call a *protocol*: playing 'fair', each player P_i can guarantee that $\mu_i(W_i) \geq \mu_i(W_j)$ for all $j \in [m]$ even if the other players do not consistently stick to their measures.

References

[1] S. J. Brams and A. D. Taylor, An envy-free cake division protocol, *Amer. Math. Monthly* **102** (1995) 9–18.

[2] S. Even and A. Paz, A note on cake-cutting, *Discrete Appl. Math.* **7** (1984) 285–286.

[3] G. Gamov and M. Stern, *Puzzle-Math*, Viking, New York, 1958.

[4] J. M. Robertson and W. A. Webb, Near exact and envy-free cake division, *Ars Combin.* **45** (1997) 97–108.

Department of Pure Mathematics and Mathematical Statistics, Cambridge University, Cambridge CB2 1SB, UK

O.Pikhurko@dpmms.cam.ac.uk