CUTTING AND PASTING

LIBERTÉ, ÉGALITÉ, HOMOLOGIE!
1. Heegaard splittings
   – Making 3-manifolds from solid handlebodies

2. Surface homeomorphisms
   – Gluing manifolds together along their boundaries

3. Surgery
   – Cutting and pasting

4. Homology spheres
   – If it looks like a sphere... it might not be.
1. Heegaard Splittings

\textit{n-manifold}: Compact, connected, Hausdorff topological space $M^n$, each point of which has a neighbourhood homeomorphic to $\mathbb{R}^n$.

\ldots with boundary: Also allow neighbourhoods homeomorphic to $\mathbb{R}^n_+$. \\
\textit{boundary}, $\partial M$: The bit consisting of points with neighbourhoods homeomorphic to $\mathbb{R}^n_+$.

\textbf{Genus-}n \textbf{handlebody}: Compact subset of $\mathbb{R}^3$ bounded by a genus-n surface (a 2-sphere with n hollow handles).

A genus-3 handlebody
Now:
1. Take two identical copies $M_1, M_2$ of the genus-$n$ handlebody.
2. Choose a homeomorphism $f : \partial M_1 \to \partial M_2$.
3. Form the quotient space $M = M_1 \amalg_f M_2$: Take the disjoint union $M_1 \amalg M_2$ and identify $x \in M_1$ with its image $f(x) \in M_2$.

This is a **Heegaard splitting** (of genus $n$) of the 3-manifold $M$.

**Theorem**
A 3-manifold formed in this way is orientable. Furthermore, any orientable 3-manifold can be presented thusly.
**EXAMPLES**

1. Genus-0: The 3-sphere $S^3$.
   Two copies of $B^3$ glued together along their boundaries ($\cong S^2$).
   Analogous to $S^1 = D^1 \cup D^1$
   and $S^2 = D^2 \cup D^2$.
   The ‘North’ and ‘South’ poles of $S^3$ are the centres of the $B^3$'s

2. Genus-1: The 3-sphere $S^3$ (again).
   Can also split $S^3$ as two solid tori ($D^2 \times S^1$):
3. $S^2 \times S^1$.
   Glue $\alpha_1$ to $\alpha_2$ and $\beta_1$ to $\beta_2$.

4. Projective space $\mathbb{RP}^3$.
   Glue each meridian $\alpha_i$ to a $(1, 2)$ torus knot on the surface of the other solid torus.

5. Lens spaces $L_{p,q}$.
   Glue each meridian $\alpha_1$ and $\alpha_2$ to a $(q, p)$ torus knot on the surface of the other torus. This is one construction of the lens space $L_{p,q}$.
   In particular,
   $L_{1,q} \cong S^3$,
   $L_{0,1} \cong S^2 \times S^1$,
   and $L_{2,1} \cong \mathbb{RP}^3$.
   Also, $L_{p,q} \cong L_{p,q'}$ iff $\pm q' \equiv q^{\pm 1} \pmod{p}$,
   and $L_{p,q} \cong L_{p,q'}$ iff $q \equiv m^2 q' \pmod{p}$.
   For example: $L_{7,1}$ and $L_{7,2}$ have the same homotopy type, but are not homeomorphic.
HOMEOMORPHISMS OF SURFACES
Heegaard splittings of a given genus are determined by the gluing homeomorphism. Want a description of such in terms of suitable elementary operations.

DEHN TWISTS
Cut the surface around a meridional curve, twist, and glue back together again.

THEOREM (DEHN-LICKORISH)
Any orientation-preserving homeomorphism of an oriented 2-manifold (without boundary) is isotopic to a composition of Dehn twists.
**Corollary**

Any orientable 3-manifold can be constructed by cutting out a collection of unknotted solid tori from $S^3$ and gluing them back in along different boundary homeomorphisms.

**Corollary (Rokhlin’s Theorem)**

Every orientable 3-manifold (without boundary) is the boundary of a 4-manifold. That is, $\Omega_3 \cong 0$. 
RATIONAL SURGERY
Take $S^3$, cut out an unknotted solid torus with meridian $\alpha$ and longitude $\beta$. Then glue it back in by identifying $\alpha$ with the curve $p\alpha + q\beta$, where $p$ and $q$ are coprime (this is a $(q, p)$ torus knot).

This surgery is determined completely by the rational number $r = \frac{p}{q}$, which we call the **framing index** of the unknotted torus.

EXAMPLES

1. $\bigcirc^0 \cong S^1 \times S^2$. This is a **torus switch**

2. $\bigcirc_{\frac{p}{q}} \cong L_{p, q}$.

3. $\bigcirc^{\pm \frac{1}{n}} \cong S^3$.

4. $\bigcirc^r = \bigcirc^{\pm n + \frac{1}{r}}$


LINKING NUMBERS

To generalise to nontrivial knots, we need to be a bit more careful when choosing the longitude.

Given two curves $J$ and $K$ in $S^3$, define their linking number $\text{lk}(J, K)$ to be the sum, over all the crossings $\tau$, of $\varepsilon(\tau)$:

Crossings which don’t involve two different components of the link have $\varepsilon = 0$.

$\text{lk}(J, K) = \text{lk}(K, J)$.

If $\text{lk}(J, K) \neq 0$ then $J$ and $K$ are linked. The converse isn’t true in general, though.

The linking number is invariant under the Reidemeister moves: It’s independent of the isotopy class of the link.
Now, given a knot $J$, choose a meridian $\alpha$ on its tubular neighbourhood such that $\text{lk}(\alpha, J) = 1$, and a longitude $\beta$ which is codirected with $J$ such that $\text{lk}(\beta, J) = 0$.

**INTEGER SURGERY**

We can now do rational surgery on nontrivial links. It turns out, though, that integer surgery is enough:

**THEOREM**

Any compact, orientable 3-manifold without boundary can be obtained by integer surgery on a link in $S^3$. 


**EQUIVALENT SURGERIES**

Surgery on $S^3$ along different framed links can produce homeomorphic manifolds. Two such surgeries are said to be **equivalent**.

**THE KIRBY CALCULUS**

An $O_1$-move consists of adding or deleting an unlinked trivial knot with framing $\pm 1$:

An $O_2$-move consists of a **handle-slide**:
Theorem (Kirby)
Two links in $S^3$ with integer framings produce the same 3-manifold iff they can be obtained from each other by a finite sequence of Kirby moves and isotopies.

Fenn-Rourke moves
A Fenn-Rourke move is as follows:

(If the circle has framing $-1$, then the kinks go the other way.)

Theorem (Fenn-Rourke)
A framed link $L_1$ can be transformed by Kirby moves into the framed link $L_2$ iff this can be done by Fenn-Rourke moves.
THE FUNDAMENTAL GROUP $\pi_1$

The **fundamental group** of a topological space $X$, denoted $\pi_1(X)$ is essentially a way of counting the (1-dimensional) holes in $X$. Its elements are the homotopy classes of based loops (maps $S^1 \to X$) in $X$, with the multiplication operation being given by concatenation and the identity being the loop which can be shrunk down to the basepoint.

For example: $\pi_1(B^3) \cong 0$, because every loop can be shrunk down to the basepoint. But $\pi_1(S^1 \times D^2) \cong \mathbb{Z}$, homotopy classes of loops being determined by the number of times they wind around the central hole. And $\pi_1(L_{p,q}) \cong \mathbb{Z}_p$.

HOMOLOGY GROUPS $H_n$

The **homology groups** $H_n(X)$ are another way of counting the $n$-dimensional holes in $X$.

In particular, part of Hurewicz’ theorem says that, if $X$ is path-connected:

**PROPOSITION**

$H_1(X) \cong \pi_1(X)/[\pi_1(X), \pi_1(X)]$.

That is, the first homology of $X$ is the same as the abelianisation of the fundamental group.
**Homology 3-spheres**

A **homology 3-sphere** is a compact, path-connected 3-manifold $M^3$ (without boundary), which has the same series of homology groups as $S^3$:

$$H_0(M) \cong H_3(M) \cong \mathbb{Z},$$
$$H_1(M) \cong H_2(M) \cong 0.$$

Or (by Poincaré duality and the UCT):

$H_1(M)$ is trivial.

Or (by the above fragment of Hurewicz’ theorem):

$\pi_1(M)$ coincides with its commutator subgroup

$$[\pi_1(M), \pi_1(M)] = \{aba^{-1}b^{-1} \mid a, b \in \pi_1(M)\}$$

**Conjecture (Poincaré)**

Every homology 3-sphere is homeomorphic to $S^3$.
This is false, leading Poincaré to suggest:

**Conjecture (Poincaré)**

Every *homotopy* 3-sphere is homeomorphic to $S^3$.

**Poincaré’s Homology 3-Sphere**

This is a 3-manifold $P^3$ with trivial $H_1$ but nontrivial $\pi_1$ (and is hence not homeomorphic to $S^3$).

Many different constructions...

(cf. Kirby and Scharlemann: *Eight faces of the Poincaré homology 3-sphere*)

**Dodecahedral Space**

Take a solid dodecahedron and identify opposite faces with a $\frac{2\pi}{10}$ twist.
Surgery on the trefoil
Do surgery on $S^3$ along the right trefoil $K$ with framing $+1$.

This gives a manifold $P$ which isn’t homeomorphic to $S^3$, because $\pi_1(P)$ is nontrivial.

Calculation of $\pi_1(P)$
First, calculate the fundamental group of the complement $S^3 \setminus K$. This is generated by three loops $x, y, z$
subject to the relations $xy = yz = zx$:

\[
\begin{array}{c}
\begin{array}{c}
\text{z} \\
\text{x} \\
\text{y}
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
\text{y} \\
\text{x} \\
\text{z}
\end{array}
\end{array}
\]

giving a presentation

\[
\langle x, y \mid xyx = yxy \rangle
\]

When we glue in the solid torus, we attach its meridional disk to the longitude of the tubular neighbourhood of the (removed) trefoil, with framing +1.

Thus, $\pi_1(P)$ is the quotient of $\pi_1(S^3 \setminus K)$ obtained by killing the word $x^{-2}yxz = x^{-2}yx^2yx^{-1}$ corresponding to this longitude:

\[
\pi_1(P) = I = \langle x, y \mid xyx = yxy, yx^2y = x^3 \rangle
\]
By substituting $a = x, b = xy$, we get the neater form:

$$I \cong \langle a, b \mid a^5 = b^3 = (ba)^2 \rangle$$

This group (the **binary icosahedral group**) is non-trivial (it has order 120 and is the isometry group of the icosahedron), hence $P \not\cong S^3$, but it has trivial abelianisation.

**Surgery on the Whitehead Link**

![Diagram of the Whitehead Link]

**Surgery on the Borromean Rings**

![Diagram of the Borromean Rings]