The Millennium Prize Problems

EightSquaredCon 2013

Easter 2013

The Riemann Hypothesis

Problem

Prove that all nontrivial zeroes of Riemann's zeta function ζ have real part $\frac{1}{2}$.

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \cdots$$

Euler's product formula

$$\zeta(s) = \prod_{p \text{ prime}} \frac{1}{1 - p^{-s}} = \frac{1}{1 - \frac{1}{2^s}} \times \frac{1}{1 - \frac{1}{3^s}} \times \frac{1}{1 - \frac{1}{5^s}} \times \cdots$$

Riemann's functional equation

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s) \zeta(1-s)$$

The Riemann Hypothesis

Let $\pi(x)$ be the number of prime numbers less than or equal to x:

$$\pi(10) = 4,\ \pi(100) = 25,\ \pi(1000) = 168,\ \pi(10\,000) = 1229, \ldots$$

The Prime Number Theorem

$$\frac{x}{\ln(x)} \longrightarrow \pi(x) \qquad \text{as } x \longrightarrow \infty$$

The logarithmic integral

$$\mathrm{li}(x) = \int_0^x \frac{1}{t}\,dt$$

If the Riemann Hypothesis is true, then

$$\left|\pi(x) - \mathrm{li}(x)\right| < \frac{\ln(x)\sqrt{x}}{8\pi}.$$

The Navier–Stokes Equations

Problem

Prove or disprove that in three space dimensions and time, given an initial velocity field, there exists a vector velocity and a scalar pressure field, which are both smooth and globally defined, that solve the Navier–Stokes equations.

Navier–Stokes equations

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\nabla p + \nu \Delta \mathbf{v} + \mathbf{f}(\mathbf{x}, t)$$

- ν is the kinematic viscosity,
- $\mathbf{f}(\mathbf{x}, t)$ is the external force,
- $\nabla = \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}$ is the gradient operator, and
- $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is the Laplacian operator.

P vs NP

Problem

Determine whether every language accepted by some nondeterministic algorithm in polynomial time is also accepted by some (deterministic) algorithm in polynomial time.

Yang–Mills Existence and Mass Gap

Problem

Prove that for any compact simple gauge group G, a non-trivial quantum Yang–Mills theory exists on $\mathbb{R}^4$ and has a mass gap $\Delta > 0$.

- Yang–Mills Theory: nonabelian quantum field theory underlying the Standard Model.
- $\mathbb{R}^4$: 4–dimensional Euclidean space.
- Gauge group G: underlying symmetry group of the theory.
  - Electromagnetic interactions: U_1
  - Weak interactions: SU_2
  - Strong interactions: SU_3
- Mass gap Δ: mass of lightest particle predicted by the theory.

Problem

Show that every Hodge class on a projective complex manifold X is a linear combination with rational coefficients of the cohomology classes of complex subvarieties of X.

$$\mathrm{Hdg}^k(X) = H^{2k}(X, \mathbb{Q}) \cap H^{k,k}(X).$$

- Projective complex manifold: manifold with some extra structure.
- Cohomology group $H^n(X; A)$: topological gadget describing the n–dimensional structure of X, counted using elements of A.
- $H^n(X) := H^n(X; \mathbb{Z})$ (count with integers).
- $H^{k,k}(X)$: subgroup of $H^{2k}(X)$ consisting of cohomology classes represented by harmonic forms of type (k, k).

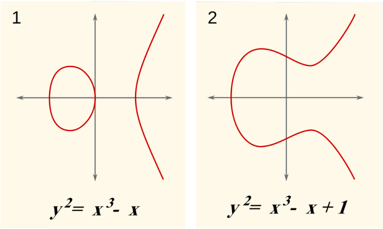
Problem

Let C be a rational elliptic curve. Show that the Taylor expansion of the Hasse L–series $L(C,s)$ at $s=1$ has the form

$$L(C,s) = c(s-1)^r + \textit{higher order terms}$$

with $c \neq 0$ and $r = \mathrm{rank}(C)$.

- Elliptic curve: $y^2 = x^3 + ax + b$



The Poincaré Conjecture

Problem

Prove that every closed, simply-connected 3–manifold is homeomorphic to the 3–sphere S^3.

Definition

An n–manifold is an object (a Hausdorff topological space) that locally "looks like" (is homeomorphic to) ordinary n–dimensional space $\mathbb{R}^n$.