THE UNIVERSITY OF WARWICK

FOURTH YEAR EXAMINATION: MOCK EXAM

ALGEBRAIC CURVES

Time Allowed: **3 hours**

Read carefully the instructions on the answer book and make sure that the particulars required are entered on each answer book.

Calculators are not needed and are not permitted in this examination.

ANSWER 4 QUESTIONS.
If you have answered more than the required 4 questions in this examination, you will only be given credit for your 4 best answers.

The numbers in the margin indicate approximately how many marks are available for each part of a question.

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1. (i) Let $A \subset K$ be a subring of a field. If $y \in K$ is integral over A, prove that the subring $A[y] \subset K$ is a finite A-module (finitely generated as A-module). **[2]**

Generalise the statement to the subring $A[y_1, \dots, y_n] \subset K$ generated by finitely many integral elements. [The proof is not required.] **[1]**

(ii) Suppose now that $A \subset B \subset K$ with the subring B a finite A-module. Prove that any $b \in B$ is integral over A. [Hint: Choose generators of B over A, and write out the A-linear map of multiplication by b as a matrix with entries in A. Argue on the determinant of b times the identity minus this matrix.] **[10]**

(iii) Deduce from (i–ii) that the sum and products of elements of K that are integral over A are again integral over A, hence that the integral closure of A in K is a subring. **[4]**

(iv) Calculate the integral closure of the ring $A = k[x, y]/(y^3 - x^8)$ in its field of fractions. **[5]**

Explain briefly how taking normalisation (integral closure) provides the nonsingular model of the curve $C \subset \mathbb{A}^2$ given by $y^3 = x^8$. **[3]**

Status: (i–iii) is bookwork; (iii) looks obvious, but depends on (ii), which is tricky to do directly. (iv) is unseen, but similar to material on the example sheets.

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2. (i) Give the definition of discrete valuation ring. Explain how it relates to the notion of nonsingular point of a curve C. Show how to define the divisor $\operatorname{div} f$ of a function $f \in k(C)^\times$, and explain what it means in terms of zeros and poles of f at points of C. **[5]**

(ii) Let $C_a \subset \mathbb{P}^2$ be a nonsingular curve of degree a in $\mathbb{P}^2$ with homogeneous coordinates x_1, x_2, x_3. Give the definition of multiplicity of intersection $\operatorname{mult}_P(C_a, L)$ of C_a with a line $L \subset \mathbb{P}^2$ at $P \in C$, and relate it to the divisor of $L/x_i \in k(C)^\times$ (for appropriate choice of x_i). **[5]**

Write $\operatorname{div} L$ for the divisor on C corresponding to $\sum \operatorname{mult}_P(C, L)$. Prove that the divisors $\operatorname{div} L$ for different L are all linearly equivalent. **[3]**

From now on let $C = C_3$ be a nonsingular plane cubic curve. You may assume as given that $g(C) = 1$, and that every line $L \subset \mathbb{P}^2$ meets C in 3 points counted with multiplicity.

(iii) Use RR to prove that any divisor D of degree 1 is linearly equivalent to P for a unique point $P \in C$. For $P_1, P_2, P_3 \in C$, give a geometric construction for the point Q that is linearly equivalent to $P_1 + P_2 - P_3$. **[4]**

(iv) Write A for the group of divisors of degree 0 modulo linear equivalence. For $O \in C$ a marked point, show that the map $P \mapsto [P - O] \in A$ defines a bijective map $C \to A$. Prove that C has a group law with O as unit element such that the sum $\sum \operatorname{div} L$ of the 3 points of $L \cap C$ is constant. **[8]**

Status: Bookwork. (iii) relates to the geometric construction of the group law.

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3. (i) Let C be a nonsingular projective curve. Define a divisor on C, linear equivalence of divisors, and the Riemann–Roch space $\mathcal{L}(C, D)$. Prove that if D_1 and D_2 are linearly equivalent then $\mathcal{L}(D_1) \cong \mathcal{L}(D_2)$. **[5]**

(ii) Prove that for a given divisor D and any $P \in C$,

$$l(D - P) = \text{either } l(D) - 1 \quad \text{or} \quad l(D).$$

If $l(D) \neq 0$, prove that the second possibility occurs for at most a finite number of points of C. **[5]**

(iii) Give the full statement of the Riemann–Roch theorem, assuming the definition of the canonical divisor class K_C, and use it to prove the following assertions:

(a) $\deg K_C = 2g - 2$ and $l(K_C) = g$. **[5]**

(b) For any integer n with $0 \leq n \leq g$, there exist n points $P_1, \ldots, P_n$ such that $l(P_1 + \cdots + P_n) = 1$. **[5]**

(c) For any integer m with $g - 2 \leq m \leq 2g - 2$, there exists a divisor D on C with $\deg D = m$ for which $l(D) = m - g$. **[5]**

Status: (i–ii) and (iii,a) is bookwork. (iii.b–c) are unseen, but follow from the methods of argument around the proof of RR.

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4. Part 1. The proof of RR used in the course was based on three main propositions. The first two of these are:

(I) A principal divisor has degree zero: $\deg(\operatorname{div} f) = 0$ for all $f \in k(C)^\times$.

(II) There exists a sequence of divisors D_n of degree tending to $+\infty$ such that the difference $\deg D_n + 1 - l(D_n)$ is bounded.

Use (I) and (II) together with the results of Question 2, (i–ii) to prove the following results:

(i) The maximum $g = \max_D\bigl\{\deg D + 1 - l(D)\bigr\}$ taken over all divisors D is well defined, so that the Riemann–Roch inequality $l(D) \geq 1 - g + \deg D$ is satisfied for every divisor D. **[5]**

(ii) With g as in (i), every divisor D of degree $\geq g$ has $l(D) > 0$, so is linearly equivalent to an effective divisor. **[2]**

(iii) There exists a divisor D of degree $g - 1$ for which $l(D) = 0$, so that the RR inequality is equality. **[4]**

(iv) $l(D) = 1 - g + \deg D$ holds for every divisor D of degree $\geq 2g - 1$. **[4]**

Part 2. Suppose that $g(C) = 2$ and $\deg D = 4$. Prove that $l(D - K_C) \neq 0$, and deduce that φ_D is not an embedding. Show that φ_D is either a generically 2-to-1 map of C to a plain conic, or maps C birational to a quartic curve $\overline{C}$ with a node or cusp as its only singularity. Explain which divisors D correspond to each case. [You may use the criteria on embeddings, and standard properties of the canonical map of a genus 2 curve.] **[10]**

Status: Part 1 is bookwork. The whole proof of RR is too long for an exam question, but it is fair to state parts of it as given, and ask for the proof of the next part. Part 2 is part of a past exam questions (that is basically too hard), and was discussed on an example sheet.

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5. (i) Let C be a nonsingular projective curve and A, B divisors on C. Prove that multiplication in $k(C)$ defines a k-bilinear map

$$\mathcal{L}(C, A) \times \mathcal{L}(C, B) \to \mathcal{L}(C, A + B).$$

(In other words, $f \in \mathcal{L}(C, A)$ and $g \in \mathcal{L}(C, B)$ implies $fg \in \mathcal{L}(C, A + B)$.) **[3]**

(ii) Let C be a nonsingular projective curve and D a divisor of degree d with $\mathcal{L}(C, D)$ of dimension 2, with basis s_1, s_2. Say what it means for the linear system $|D|$ to be a free g^1_d. **[3]**

(iii) Let $|D|$ be a free g^1_d as in (ii), and A any divisor. Determine the intersection

$$s_1 \cdot \mathcal{L}(C, A) \cap s_2 \cdot \mathcal{L}(C, A) \subset \mathcal{L}(C, A + D).$$

Deduce that the subspace $s_1 \cdot \mathcal{L}(C, A) + s_2 \cdot \mathcal{L}(C, A)$ in $\mathcal{L}(C, A + D)$ has dimension equal to $2l(A) - l(A - D)$. **[12]**

(iv) Now assume in addition that $\deg A - \deg D \geq 2g - 1$. Prove that the image of the multiplication map of (i) spans $\mathcal{L}(A + D)$. **[7]**

Status: (ii–iii) is part of the Castelnuovo free pencil trick, that was lectured. (iv) is unseen, but not difficult.
