MA4L7 Algebraic curves

Example sheet 5, Deadline Tue 19th Feb

1. The Macaulay quartic curve $\Gamma_4 \subset \mathbb{P}^3$ Review the material on the Macaulay quartic curve $\Gamma_4 \subset \mathbb{P}^3$ of (9.7). Calculate the dimension of the quotient $k[\mathcal{C}\Gamma_4]/(x,t)$. In other words, calculate $k[y,z]$ modulo the ideal given by the 4 equations (9.2) with x,t set equal to zero.

2. Regular sequences and free module Write $x \in S$ be a ring element. Let M be a finite S-module (that is finite as module). Suppose that x is a nonzero divisor of M and set $\overline{M} = M/xM$, so that

$$0 \to M \xrightarrow{x} M \to \overline{M}$$

is an exact sequence. Prove that M is free over S if and only if $\overline{M}$ is free over $\overline{S} = S/(x)$.

Do the same for graded ring S and graded module M.

Write out a detailed proof of Lemma 9.5.

3. Calculations assuming $R(C,D) = \bigoplus S(-a_i)$ Carry out the easy calculations involved in the proof of Proposition 9.4, (C).

4. RR calculations for $\mathcal{K} = \mathrm{Hom}_S(R(C,D), S(-2))$ Carry out the easy calculations involved in the proof of Proposition 9.6.

5. Plane curve with ordinary multiple points Let $\Gamma \subset \mathbb{P}^2$ be a plane curve of degree a having an ordinary multiple point of multiplicity m at Q. The normalisation (resolution of singularties) $C \to \Gamma$ has m points $P_1, \dots, P_m$ over Q, corresponding to the m tangent branches of Γ at Q.

Treat $C \to \Gamma$ as local or affine. (This means shrink Γ to an affine neighbourhood of Q, and take the inverse image of that in C, which contains all of $P_1, \dots, P_m$. Or just treat Γ as the local ring $\mathcal{O}_{\Gamma,Q}$ contained in the semilocal ring $\bigcap \mathcal{O}_{C,P_i} \subset k(C)$.)

The Brill–Noether method developed in Fulton's book asserts that forms on $\mathbb{P}^2$ of degree $n \geq a-3$ vanishing $m-1$ times at Q (that is, in the conductor ideal $\mathcal{C} = m_Q^{m-1}$) map surjectively to the RR space $\mathcal{L}(C, K_C + (n-a+3)H)$. Here H is the hyperplane section divisor, and K_C is the divisor $(a-3)H - (m-1)\sum P_i$.

Calculate the degree of all the divisors involved, and verify that the RR theorem hold for them. (That is, the equality if $n \geq a-2$, the value of g if $n = a-3$, and the difference $l(D) - l(K-D)$ when $n < a-2$.)

[The point of Fulton's book is that every curve C is birational to a plane curve Γ with ordinary multiple points Q_i of order m_i, and the adjoint curves of degree $n+a-3$ give an exact description of for the RR spaces of $K_C + aH$. This is the Brill–Noether method of proof of RR. At the same time, it provides a vast catalogue of examples of constructions of curves.]

5. Conductor ideal [Harder] As in Q.**??** $\Gamma \subset \mathbb{P}^2$ be a plane curve of degree a having an ordinary multiple point of multiplicity m at Q.

The conductor of the normalisation

$$\mathcal{C} = [k[Ga] : k[C]] = \mathrm{Ann}(k[C]/k[Ga]) = \mathrm{Hom}(\mathcal{O}_C, \mathcal{O}_\Gamma)$$

is the ideal of functions f in $k[\Gamma]$ so that $f \cdot k[C] \subset k[\Gamma]$. By considering the quotient rings $\mathcal{O}_{\Gamma,Q}/m_Q^N$ and $\bigoplus \mathcal{O}_{C,P_i}/m_{P_i}^N$ for any $N \geq m$, prove that $\mathcal{C} = m_Q^{m-1}$.

[Hint: This is all finite dimensional linear algebra. The quotient ring $\mathcal{O}_{\Gamma,Q}/m_Q^N$ is isomorphic to the polynomial ring $k[x_1, x_2]/(x_1, x_2)^N$, so Taylor series in 2 variables up to degree N. Similarly each $\mathcal{O}_{C,P_i}/m_{P_i}^N$ is Taylor series in 1 variable up to degree N.]