13. (i) Suppose that k is a subfield of $\mathbb{C}$, and let $C \subset \mathbb{P}^2_k$ be a nonsingular cubic curve. Show that C meets any line L of $\mathbb{P}^2$ in 3 points "counted with multiplicities", and describe briefly how multiple intersections of L and C relate to geometric notions of tangency and inflexion.

(ii) Suppose that $C : (Y^2Z = X^3 + aXZ^2 + bZ^3)$, and let $O = (0, 1, 0)$; show how to construct a group law on C with O as origin; your construction must be everywhere defined, and you should explain the construction of the inverse, but need not prove the group axioms.

(iii) Suppose C is given in affine coordinates by $C : (y^2 = x^3 + x^2 + 3x - 1)$; write down the tangent line to C at $P = (1, 2) \in C$, and show that P generates a subgroup of C of order 3.

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or the same with (iii) replaced by

(iii′) Suppose C is given in affine coordinates by $C : (y^2 = x^3 + 3x)$ and consider the points $P = (0, 0)$ and $Q = (1, -2) \in \mathbb{C}$. Show that $P + Q = R = (3, 6)$ in the group law of C, and determine the point $Q + R$.

(iii′) Suppose C is given in affine coordinates by $C : (y^2 = x^3 + 1)$, and let $P = (2, 3)$ and $Q = (0, 1) \in C$. Calculate $P + Q$ in the group law. Calculate the tangent line to C at P, and using this, show that $2P = Q$. Prove that P generates a subgroup of C of order 6. [Hint: The tangent line to C at Q is $(y = 1)$.]

(iii′) Suppose C is given in affine coordinates by $C : (Y^2 = X^3 + 4X)$, and as usual O is chosen to be the point at infinity. Show that the tangent line to C at $P = (2, 4)$ passes through $(0, 0)$, and deduce that P is a point of order 4 in the group law.