Algebraic geometry MA4A5

Homework on Chapter 1

1.A [UAG], Ex. 1.1. Parametrise the circle $x^2 + y^2 = 5$ by considering the variable line through $(2, 1)$ and hence find all rational solutions of $x^2+y^2 = 5$.

1.B Section of a past exam paper. Let C be the conic $C : (XZ = Y^2) \subset \mathbf{P}^2$ and $P = (1, 0, 0)$, $Q_1 = (1, 1, 1)$, $Q_2 = (0, 0, 1)$. Using the usual parametrisation, write down a conic C' such that $C \cap C' = 2P + Q_1 + Q_2$ (see [UAG], (1.12)).

Find all the singular conics in the pencil $\lambda C + \mu C'$; describe their intersection with C in geometric terms.

1.C [UAG], (1.7) says that a nondegenerate conic can be parametrised. Discuss the converse: a map $\mathbf{P}^1 \to \mathbf{P}^2$ defined by

$$(U, V) \mapsto (a_0(U, V), a_1(U, V), a_2(U, V)),$$

where a_0, a_1, a_2 are linearly independent quadratic forms in U, V has image a plane conic. Determine the image if

$$a_0 = U^2 - 2V^2, \quad a_1 = UV, \quad a_2 = U^2 + V^2.$$

1.D Section of a past exam paper. (Omit the bookwork part.)

Show that if $P_1, \dots, P_4$ are 4 points of $\mathbf{P}^2$ such that no 3 are collinear then there is exactly a pencil $\{\lambda Q_1 + \mu Q_2\}$ of conics in $\mathbf{P}^2$ through them. Determine all the conics through the 5 points

$$(1, 0, 0) \quad (0, 1, 0) \quad (0, 1, 1) \quad (1, 1, 1) \quad (2, 1, 3).$$

[Hint: You should suspect a trick. The examiner has some labour saving devices.]