ORBIFOLD RIEMANN–ROCH AND HILBERT SERIES

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Abstract. We consider quasismooth polarized n-folds (X, D); then X only has cyclic orbifold points (that is, cyclic quotient singularities). We give an explicit orbifold Riemann–Roch formula for the Hilbert series of (X, D), under the extra assumptions that X is projectively Gorenstein with only isolated orbifold points. Our formula is a sum of terms each of which is integral and Gorenstein symmetric of the same canonical weight; the orbifold terms are called *ice cream functions*. This form of the Hilbert series is particularly useful for computer algebra, and we illustrate it on examples of K3 surfaces and Calabi–Yau 3-folds, and show how to use it construct new families of projective n-folds.

These results apply also with higher dimensional orbifold strata (see [6] and [19]), although the correct statements are considerably trickier. We expect to return to this in future publications.

Contents

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1. Introduction

Reid [YPG] introduced Riemann–Roch (RR) formulas for polarized orbifolds (X, D) with isolated orbifold locus, of the form

$$\chi(X, \mathcal{O}_X(D)) = (\text{RR-type expression in } D) + \sum_{P \in \mathcal{B}} c_P(D), \tag{1.1}$$

where the $c_P(D)$ are certain fractional contributions from the orbifold points $\mathcal{B}$, depending only on the local type of (X, D). The orbifold RR formula of [YPG] has found numerous subsequent extensions and applications; see for example Iano-Fletcher [11], Brown, Altınok and Reid [2], Buckley and Szendrői [6], Chen, Chen and Chen [8] and Kawakita [13], and we expect these ideas to be equally applicable in the study of higher dimensional varieties.

A general RR formula for abstract orbifolds was first proved by Kawasaki [14] by analytic tools. Toen [17] gave another proof using the algebraic methods of Deligne–Mumford stacks. However, at present, how to use these abstract results in practice to compute the dimension of RR spaces is not well understood. Toen's result was applied to weighted projective spaces by Nironi [15] and to twisted curves by Abramovich and Vistoli [1]. Our proof, like that of [YPG], is based on a reduction to Atiyah–Singer and Atiyah–Segal equivariant Riemann–Roch [3], [4].

Let D be an ample $\mathbb{Q}$-Cartier divisor on a normal projective n-fold X (we usually work over $\mathbb{C}$). The finite dimensional vector spaces $H^0(X, \mathcal{O}_X(mD))$ fit together as a finitely generated graded ring

$$R(X, D) = \bigoplus_{m \geq 0} H^0(X, \mathcal{O}_X(mD)), \tag{1.2}$$

with $X \cong \operatorname{Proj} R(X, D)$ and the divisorial sheaf $\mathcal{O}_X(mD)$ equal to the character sheaf $\mathcal{O}_X(m)$ of the Proj. A surjection from a graded polynomial ring

$$k[x_0, \dots, x_N] \twoheadrightarrow R(X, D) \quad \text{with } \operatorname{wt} x_i = a_i \tag{1.3}$$

corresponds to an embedding

$$i \colon X \cong \operatorname{Proj} R(X, D) \hookrightarrow \mathbb{P}(a_0, \dots, a_N) \tag{1.4}$$

of X into a weighted projective space.

The Hilbert function $m \mapsto P_m(X, D) = h^0(X, \mathcal{O}_X(mD))$ and the *Hilbert series* $P_X(t) = \sum_{m \geq 0} P_m t^m$ encode the numerical data of $R(X, D)$; it is known that $\prod(1 - t^{a_i}) \cdot P_X(t)$ is a polynomial where, as above, the a_i are the weights of the generators. The multiplicative group $\mathbb{G}_m$ ($= \mathbb{C}^\times$ if the ground field is $\mathbb{C}$) has a standard action on the graded ring $R(X, D)$, with $\lambda \in \mathbb{C}^\times$ multiplying R_m by λ^m; our aim is a *character formula* for the Hilbert series.

1.1. **The main result.** For a quasismooth projectively Gorenstein orbifold (X, D) with isolated orbifold points, Theorem 1.1 parses the Hilbert series of (X, D) into simple pieces, each of which is integral and Gorenstein symmetric of the same degree k_X. We call the orbifold contributions *ice cream functions*. The result expresses $P_X(t)$ in a closed form that can be calculated readily as a few lines of computer algebra (see Section 7). See 1.3 for a reminder and explanation of the definitions.

Theorem 1.1. *Let (X, D) be a projectively Gorenstein quasismooth orbifold X of dimension $n \geq 2$ with only isolated orbifold points, that we write*

$$\mathcal{B} = \left\{Q \text{ of type } \tfrac{1}{r}(b_1, \dots, b_n)\right\}.$$

Then the Hilbert series of X is

$$P_X(t) = P_I(t) + \sum_{Q \in \mathcal{B}} P_{\mathrm{orb}}(Q, k_X), \tag{1.5}$$

where

(i) *the initial term has the form $P_I(t) = \frac{A(t)}{(1-t)^{n+1}}$, where $A(t)$ is the unique integral palindromic (Gorenstein symmetric) Laurent polynomial of degree $c = k_X + n + 1$ (the* coindex*) such that $P_I(t)$ equals the series $P_X(t)$ up to and including degree $\left\lfloor \frac{c}{2} \right\rfloor$. If $c < 0$ then $P_I = 0$.*

(ii) *Each orbifold term for $Q \in \mathcal{B}$ of type $\frac{1}{r}(b_1, \dots, b_n)$ is of the form $P_{\mathrm{orb}}(Q, k_X) = \frac{B(t)}{(1-t)^n (1-t^r)}$, with*

$$B(t) = \mathrm{InvMod}\Big(\prod_{i=1}^{n} \frac{1 - t^{b_i}}{1 - t}, \frac{1 - t^r}{1 - t}, \left\lfloor \frac{c}{2} \right\rfloor + 1\Big) \tag{1.6}$$

the unique Laurent polynomial supported in $\left[\left\lfloor \frac{c}{2} \right\rfloor + 1, \left\lfloor \frac{c}{2} \right\rfloor + r - 1\right]$ equal to the inverse of $\prod_{i=1}^{n} \frac{1-t^{b_i}}{1-t}$ modulo $\frac{1-t^r}{1-t}$; $B(t)$ has integral coefficients and is Gorenstein symmetric of degree $k_X + n + r$.

Example 1.2. Consider the general hypersurface $X_{10} \subset \mathbb{P}(1, 1, 2, 2, 3)$ with coordinates x_1, x_2, y_1, y_2, z. Then X_{10} is a 3-fold with $5 \times \frac{1}{2}(1, 1, 1)$ orbifold points along $\mathbb{P}^1 \langle y_1, y_2 \rangle$ and a $\frac{1}{3}(1, 2, 2)$ point at $P_z = (0, 0, 0, 0, 1)$. It has canonical weight $k_X = 1$ and coindex $c = k_X + n + 1 = 5$. The Hilbert series is as follows: the initial term

$$P_I = \frac{1 - 2t + 3t^2 + 3t^3 - 2t^4 + t^5}{(1 - t)^4} = 1 + t + \frac{t + t^2}{(1 - t)^2} + 2\frac{t^2 + t^3}{(1 - t)^4}, \tag{1.7}$$

handles $P_1 = 2$, $P_2 = 5$. The orbifold terms

$$P_{\mathrm{orb}}(\tfrac{1}{2}(1, 1, 1), 1) = \tfrac{-t^3}{(1-t)^3(1-t^2)}, \quad P_{\mathrm{orb}}(\tfrac{1}{3}(1, 2, 2), 1) = \tfrac{-t^3-t^4}{(1-t)^3(1-t^3)} \tag{1.8}$$

take care of the periodicity, giving

$$P_I + 5 \times P_{\mathrm{orb}}(\tfrac{1}{2}(1, 1, 1), 1) + P_{\mathrm{orb}}(\tfrac{1}{3}(1, 2, 2), 1) = \frac{1 - t^{10}}{(1 - t)^2(1 - t^2)^2(1 - t^3)}.$$

Here the numerator of P_I is palindromic of degree $c = 5$, so that P_I is Gorenstein symmetric of degree 1. The two P_{orb} terms are also integral and Gorenstein symmetric of degree 1, and they start with t^3, so do not affect the first two plurigenera P_1 and P_2.

Caution: The initial term P_I is designed to handle the first plurigenera $P_1, \dots, P_{\lfloor \frac{c}{2} \rfloor}$, but is definitely not the *leading term* of the Hilbert function controlling the order of growth of the plurigenera: in this example X_{10} is a canonical 3-fold with $K_X = \mathcal{O}(1)$, of degree $K_X^3 = \frac{10}{2\times2\times3} = \frac{5}{6}$, whereas P_I on its own would correspond to degree $K^3 = 4$ (as one reads from $\mathrm{Num}(1) =$ sum of the coefficients of its numerator). In our formula, the orbifold points contribute to the global order of growth of the plurigenera, in this case $5 \times -\frac{1}{2}$ and $-\frac{2}{3}$.

1.2. **Plan of the paper.** Section 2 defines ice cream functions as inverse polynomials modulo $1 + t + \cdots + t^{r-1}$; they contain the same information as Dedekind sums. Section 3 relates the new viewpoint of this paper to formulas currently in use for the Hilbert series of K3 surfaces, Fano 3-folds and canonical 3-folds. Section 4 takes up the results of Buckley's thesis [6] on orbifold RR for polarized Calabi–Yau 3-folds, and parses their Hilbert series into ice cream functions. Section 5 deals with the existence of the RR formula for n-folds with isolated orbifold points and its precise shape, as a preliminary to the proof of the main theorem in Section 6. Section 7 contains pseudocode algorithms for the ice cream functions appearing throughout the paper.

1.3. **Definitions and notation.** We work over an algebraically closed field k of characteristic zero. A *Weil divisor* on a normal variety X is a formal linear combination of prime divisors with integer coefficients. A Weil divisor D is $\mathbb{Q}$-*Cartier* if mD is Cartier for some integer $m > 0$.

A *cyclic orbifold point* or *cyclic quotient singularity* of type $\frac{1}{r}(b_1, \dots, b_n)$ is the quotient $\pi\colon \mathbb{A}^n \to \mathbb{A}^n/\mu_r$, where μ_r acts on $\mathbb{A}^n$ by

$$\mu_r \ni \varepsilon\colon (x_1, \dots, x_n) \mapsto (\varepsilon^{b_1} x_1, \dots, \varepsilon^{b_n} x_n). \tag{1.9}$$

We usually assume that no factor of r divides all the b_i, which is equivalent to the μ_r-action being effective; the orbifold point is isolated if and only if all the b_i are coprime to r. The sheaf $\pi_*\mathcal{O}_{\mathbb{A}^n}$ decomposes as a direct sum of eigensheaves

$$\mathcal{L}_i = \left\{ f \;\middle|\; \varepsilon(f) = \varepsilon^i \cdot f \text{ for all } \varepsilon \in \mu_r \right\} \quad \text{for } i \in \mathbb{Z}/r = \mathrm{Hom}(\mu_r, \mathbb{G}_m), \tag{1.10}$$

which are divisorial sheaves.

The notation $\frac{1}{r}(b_1, \dots, b_n)$ refers to *polarized* orbifold points; in other words, $\mathcal{O}_X(1) = \mathcal{O}_X(D)$, and the orbinates x_j of degree b_j modulo r are local sections of $\mathcal{O}_X(b_j)$. This is type ${}_1\big(\frac{1}{r}(b_1, \dots, b_n)\big)$ in the terminology of [YPG], Definition 8.3.

A polarized variety (X, D) is *quasismooth* if its affine cone $\mathcal{C}_X = \mathrm{Spec}\, R(X, D)$ is nonsingular outside the origin. In this case, the orbifold points of X

arise from the orbits of the action of the multiplicative group $\mathbb{G}_m$ that are pointwise fixed by a nontrivial isotropy group $\mu_r \subset \mathbb{G}_m$, necessarily the cyclic subgroup of rth roots of unity for some r. In terms of X itself, quasismooth holds if and only if X has locally cyclic quotient singularities $\frac{1}{r}(b_1, \dots, b_n)$ and the given Weil divisor D generates the local class group $\mathbb{Z}/r = \mathrm{Hom}(\mu_r, \mathbb{G}_m)$. Then the local index one cyclic cover defined by a local identification $\mathcal{O}_X(mD) \cong \mathcal{O}_X$ is nonsingular.

Our assumption that X is normal and polarized by a $\mathbb{Q}$-Cartier Weil divisor D (so that $\mathcal{O}_X(i) \cong \mathcal{O}_X(iD)$ for all i) implies that X has no orbifold behaviour in codimension 0 or 1, or is *well formed* in the terminology of [11]. This assumption is good here because we work with n-folds for $n \geq 2$ with isolated orbifold locus. It means that the orbifold X as a scheme already knows its local orbifold cover (the universal cover of $X \setminus \operatorname{Sing} X$), which simplifies the treatment, allowing us to circumvent the language of stacks and the graded structure sheaf $\bigoplus_{i\in\mathbb{Z}} \mathcal{O}_X(i)$ (cf. Canonaco [7]). Some of our examples involve fractional divisors on curves, and we leave the elementary treatment of the graded structure sheaf $\bigoplus_{i\in\mathbb{Z}} \mathcal{O}_X(i)$ in this case to the conscientious reader.[1]

A polarized variety (X, D) is *projectively Gorenstein* if its affine cone or the corresponding graded ring $R(X, D)$ is Gorenstein; then $\omega_X \cong \mathcal{O}_X(k_X D)$ for some $k_X \in \mathbb{Z}$, called the *canonical weight* of (X, D).

Lemma 1.3. *The Hilbert series $P_R(t)$ of a graded Gorenstein ring R satisfies the functional equation*

$$P(t) = (-1)^{n+1} t^{k_R} P(1/t). \tag{1.11}$$

Here k_R is the canonical weight of R (that is, $\omega_R = R(k_R)$).

We refer to property (1.11) as *Gorenstein symmetry* of degree k_X.

Proof. This follows from duality: R is a quotient of a weighted polynomial ring $A = k[x_1, \dots, x_N]$ with $\operatorname{wt} x_i = a_i$. A minimal free resolution

$$R \leftarrow F_0 \leftarrow F_1 \leftarrow \cdots \leftarrow F_\gamma \leftarrow 0, \tag{1.12}$$

has γ equal to the codimension $N - \dim R$, and $F_\gamma = A(-\alpha)$ is the free module of rank one and degree $-\alpha$ where $\alpha = k_R + \sum a_i$ is the *adjunction number* for $X = \operatorname{Proj} R \subset \mathbb{P}(a_1, \dots, a_n)$. Duality gives $F_{\gamma-i} \cong \mathrm{Hom}_A(F_i, F_\gamma)$ so that, over the denominator $\prod(1-t^{a_i})$ corresponding to $A = k[x_1, \dots, x_N]$, the numerator of the Hilbert series is a sum of terms $t^d + (-1)^\gamma t^{\alpha-d}$. $\square$

For quasismooth X, the statement corresponds to Serre duality. However, the proof only uses the definition and basic properties of Gorenstein graded rings, without any assumptions on the singularities of $\operatorname{Spec} R$ or $\operatorname{Proj} R$.

Following Mukai [16], we write $c = k_X + n + 1$ for the *coindex* of (X, D). By the adjunction formula, the coindex is invariant under passing to a hyperplane section of degree 1. For nonsingular varieties, we have:

[1] See Demazure [9] or Watanabe [18]; the latter also treats the graded dualising sheaf for fractional divisors.

Example 1.4.

$$\begin{array}{rl}
\text{projective space } \mathbb{P}^n & \text{has coindex } 0; \\
\text{a quadric } Q \subset \mathbb{P}^{n+1} & \text{has coindex } 1; \\
\text{an elliptic curve, del Pezzo surface} & \\
\text{or Fano 3-fold of coindex 2} & \text{has coindex } 2; \\
\text{a canonical curve, K3 surface} & \\
\text{or anticanonical Fano 3-fold} & \text{has coindex } 3; \\
\text{a canonical surface, Calabi–Yau 3-fold} & \\
\text{or anticanonical Fano 4-fold} & \text{has coindex } 4.
\end{array}$$

All our concrete examples are subvarieties in weighted projective spaces. See Iano-Fletcher [11] for definitions and properties.

2. Ice cream functions

2.1. **Fun calculation.** "Income $\frac{3}{7}$ per day means ice cream on Wednesdays, Fridays and Sundays". Consider the step function $i \mapsto \left\lfloor \frac{3i}{7} \right\rfloor$, where $\lfloor\ \rfloor$ denotes the rounddown or integral part. As Hilbert series, it gives

$$P(t) = \sum_{i \geq 0} \left\lfloor \frac{3i}{7} \right\rfloor t^i = 0 + 0t + 0t^2 + t^3 + t^4 + 2t^5 + 2t^6 + 3t^7 + \cdots, \tag{2.1}$$

with closed form

$$P(t) = \frac{t^3 + t^5 + t^7}{(1-t)(1-t^7)}. \tag{2.2}$$

Indeed, $\left\lfloor \frac{3i}{7} \right\rfloor$ increments by 1 when $i = 0, 3, 5$ modulo 7, so that

$$(1-t)P(t) = t^3 + t^5 + t^7 + t^{10} + \cdots \tag{2.3}$$

is the sum over the jumps, that repeat weekly. Multiplying (2.3) by $1 - t^7$ cuts the series down to the first week's ice cream ration:

$$(1-t)(1-t^7)P(t) = t^3 + t^5 + t^7. \tag{2.4}$$

The numerator $t^3 + t^5 + t^7$ can be seen as the

inverse of $\dfrac{1-t^5}{1-t} = 1+t+t^2+t^3+t^4 \mod \dfrac{1-t^7}{1-t} = 1+t+t^2+t^3+t^4+t^5+t^6.$

Indeed, long multiplication gives

$$\begin{aligned}
(2.5)\quad (1 + t + t^2 + t^3 + t^4) \times (t^3 + t^5 + t^7) = & \\
t^3 + t^4 + t^5 + t^6 + t^7 + & \\
t^5 + t^6 + t^7 + t^8 + t^9 & \\
t^7 + t^8 + t^9 + t^{10} + t^{11} & \\
= t^3 + t^4 + 2t^5 + 2t^6 + 3t^7 + 2t^8 + 2t^9 + t^{10} + t^{11} & \\
\equiv 3 + 2t + 2t^2 + 2t^3 + 2t^4 + 2t^5 + 2t^6 \equiv 1, &
\end{aligned}$$

where $\equiv$ denotes equivalence modulo $\frac{1-t^7}{1-t}$; here $5 = \operatorname{InvMod}(3,7)$ is the inverse of 3 modulo 7. The product in (2.5) has $5 \times 3 = 15 \equiv 1 \mod 7$ terms that distribute themselves equitably among the 7 congruence classes, except that t^7 appears once for each of the 3 terms in the second factor. The same calculation works on replacing $\frac{3}{7}$ by a general reduced fraction $\frac{a}{r}$, leading to the ice cream function $\operatorname{InvMod}(\frac{1-t^b}{1-t}, \frac{1-t^r}{1-t})$ with b the inverse of a mod r.

There are several other meaningful expressions for $P(t)$. Under $\equiv$, the bounty $t^3+t^5+t^7$ can be viewed as famine $-t-t^2-t^4-t^6$ "no ice cream on Mondays, Tuesdays, Thursdays or Saturdays". In other words,

$$P(t) = \frac{t^3+t^5+t^7}{(1-t)(1-t^7)} = \frac{t}{(1-t)^2} + \frac{-t-t^2-t^4-t^6}{(1-t)(1-t^7)}. \tag{2.6}$$

Because $t^7 \equiv 1$, we can shift the exponents of t up or down by 7:

$$\frac{t^{-4}+t^{-2}+1}{(1-t)(1-t^7)} \quad \text{or} \quad \frac{-t^{-1}-t-t^2-t^4}{(1-t)(1-t^7)} \tag{2.7}$$

so "ice cream rations from Monday before the start of term" or "famine from the previous Saturday". Of these possible shifts (as Laurent polynomials with short support), $t^{7i}(t^3+t^5+t^7)$ is Gorenstein symmetric of degree $10+14i$, and $t^{7i}(-t^{-1}-t-t^2-t^4)$ is Gorenstein symmetric of degree $3+14i$, and no other.

In "macroeconomic" terms, the order of growth is the linear function $\frac{3i}{7}$ with seasonal fractional corrections, that is,

$$P(t) = \frac{3}{7}\cdot\frac{t}{(1-t)^2} + \frac{-\frac{3}{7}t-\frac{6}{7}t^2-\frac{2}{7}t^3-\frac{5}{7}t^4-\frac{1}{7}t^5-\frac{4}{7}t^6}{1-t^7} \tag{2.8}$$

("on Mondays, we lose $\frac{3}{7}$ in small change", etc.). Notice the coefficient $\frac{1}{7}$ on Friday: 5 is the inverse of 3 modulo 7, so as we enjoy our second ice cream on Fridays, we lose $\frac{1}{7}$, the unit of small change.

The fractional divisor $\frac{3}{7}P$ on a nonsingular curve is an orbifold point of type $\frac{1}{7}(5)$, with orbinate in $\mathcal{L}_5$ having genuine pole of order two, but fractional zero of order $\frac{1}{7}$ in lost change. The same considerations apply with $\frac{3}{7}$ replaced by a general reduced fraction $\frac{a}{r}$, corresponding to the orbifold point $\frac{1}{r}(b)$ with b the inverse of a mod r. Example 2.11 and Exercise 2.12 give examples of graded rings over curves involving this type of orbifold points.

We can average out the seasonal corrections in (2.8) to sum to zero, giving

$$P(t) = \frac{3}{7}\cdot\frac{2t-1}{(1-t)^2} + \frac{\frac{3}{7}-\frac{3}{7}t^2+\frac{1}{7}t^3-\frac{2}{7}t^4+\frac{2}{7}t^5-\frac{1}{7}t^6}{1-t^7}, \tag{2.9}$$

where the coefficients $\sigma_i\big(\frac{1}{7}(5)\big) = \frac{3}{7}, 0, -\frac{3}{7}, \frac{1}{7}, \frac{2}{7}, \frac{2}{7}, -\frac{1}{7}$ are Dedekind sums (compare Exercise 2.9). Our main aim is to explain how to view the orbifold contributions P_{orb} in Theorem 1.1 as minor variations on this simple-minded material.

2.2. **The function Inverse Mod.** We start with the following basic result.

Theorem 2.1. *Fix a polynomial $F \in \mathbb{Q}[t]$ of degree d with nonzero constant term and nonzero term in t^d (say, monic).*

(I) *The quotient ring $\mathbb{Q}[t]/(F)$ is a d-dimensional vector space over $\mathbb{Q}$ and t is invertible in it, so that $\mathbb{Q}[t]/(F) = \mathbb{Q}[t, t^{-1}]/(F)$.*
(II) *Any range $[t^\gamma, \dots, t^{\gamma+d-1}]$ of d consecutive Laurent monomials maps to a $\mathbb{Q}$-basis of $\mathbb{Q}[t]/(F)$.*
(III) *If $A \in \mathbb{Q}[t]$ is coprime to F, we can write its inverse modulo F uniquely as a Laurent polynomial B with support in $[t^\gamma, \dots, t^{\gamma+d-1}]$.*

This is all trivial. The leading term of F is nonzero, so $1, t, \dots, t^{d-1}$ base $\mathbb{Q}[t]/(F)$. The constant term of F is nonzero so t is coprime to F, hence invertible modulo F. Multiplication by t is an invertible linear map, so multiplication by t^γ for any $\gamma \in \mathbb{Z}$ takes a basis to another basis. If A is coprime to F it is invertible in $\mathbb{Q}[t]/(F)$, and its inverse has a unique expression in any basis. □

Definition 2.2. We set $\mathrm{InvMod}(A, F, \gamma) = B$ with B as in (III); that is, $B \in \mathbb{Q}[t, t^{-1}]$ is the uniquely determined Laurent polynomial with support in $[t^\gamma, \dots, t^{\gamma+d-1}]$ such that $AB \equiv 1 \bmod F$; for different intervals, these are congruent modulo F, but different polynomials in general. We also write $\mathrm{InvMod}(A, F)$ with unspecified support for any inverse of A modulo F in $\mathbb{Q}[t]$.

2.3. **Dedekind sums as Inverse Mod.** We now relate Dedekind sums to the function InvMod. Fix positive integers r and $b_1, \dots, b_n$ and set

$$A = \prod_{j=1}^{n} (1 - t^{b_j}) \quad \text{and} \quad F = \frac{1 - t^r}{\mathrm{hcf}(1 - t^r, A)}. \tag{2.10}$$

The polynomial F is the monic polynomial with simple roots only at the rth roots of unity with $A(\varepsilon) \neq 0$, or equivalently $\varepsilon^{b_j} \neq 1$ for all b_j. Since we take out the hcf, A and F are coprime and Theorem 2.1, III applies to give $\mathrm{InvMod}(A, F, \gamma)$, the inverse of A modulo F with support in $[t^\gamma, \dots, t^{\gamma+d-1}]$ (here $d = \deg F$ and $\gamma \in \mathbb{Z}$ is a free choice).

The case of isolated orbifold point is when $b_1, \dots, b_n$ are coprime to r; then $\mathrm{hcf}(1 - t^r, A) = 1 - t$ and $F = 1 + t + \dots + t^{r-1}$ has degree $d = r - 1$ and roots $\varepsilon \in \mu_r \setminus \{1\}$, the nontrivial rth roots of 1. If r is prime then F is the cyclotomic polynomial, and working modulo F is essentially the same thing as setting $t = \varepsilon$ a primitive rth root of unity.[2]

Algorithm 2.3. If $\gamma \geq 0$ then $t^\gamma A$ and F are coprime polynomials; set $d = \deg F$. The Euclidean algorithm in $\mathbb{Q}[t]$ provides a unique solution to

$$t^\gamma AB + FG = 1, \tag{2.11}$$

[2]To stress that $b_1, \dots, b_n$ are not all coprime to r, we may call σ_i the ith *generalized* Dedekind sum.

with $B \in \mathbb{Q}[t]$ a polynomial of degree $< d$. Then $\operatorname{InvMod}(A, F, \gamma) = t^\gamma B$.

If $\gamma < 0$, choose some m with $mr + \gamma \geq 0$, and solve

$$t^{mr+\gamma} AB + FG = 1. \tag{2.12}$$

using the Euclidean algorithm. Then $\operatorname{InvMod}(A, F, \gamma) = t^\gamma B = t^{mr+\gamma} B/t^{mr}$. This trick works because $t^{mr} \equiv 1$ modulo F. (For more general polynomials F, one needs to calculate powers of the matrix M_t corresponding to multiplication by t in $\mathbb{Q}[t]/(F)$; in our case, $M_t^r = 1$.)

Proposition 2.4. *Consider the following $r \times r$ system of linear equations, with unknowns σ_i indexed by $i \in \mathbb{Z}/r = \operatorname{Hom}(\mu_r, \mathbb{C}^\times)$, and equations indexed by $\varepsilon \in \mu_r$.*

$$\sum_{i=0}^{r-1} \sigma_i \varepsilon^i = \begin{cases} 0 & \text{if } \varepsilon \in \mu_{b_j} \text{ for some } j, \\ \dfrac{1}{(1-\varepsilon^{-b_1}) \cdots (1-\varepsilon^{-b_n})} & \text{otherwise.} \end{cases} \tag{2.13}$$

Then (2.13) is a nondegenerate system, with unique solution the Dedekind sums

$$\sigma_i = \sigma_i(\tfrac{1}{r}(b_1, \dots, b_n)) = \frac{1}{r} \sum_{\substack{\varepsilon \in \mu_r \\ \varepsilon^{b_j} \neq 1 \forall j}} \frac{\varepsilon^i}{(1-\varepsilon^{b_1}) \cdots (1-\varepsilon^{b_n})}, \tag{2.14}$$

where the sum runs over the roots of F (that is, all the rth roots of unity ε giving nonzero denominator).

For $M = (\varepsilon^{ij})$ is a Vandermonde matrix, with inverse $N = (\frac{1}{r}\varepsilon^{-ij})$. □

Lemma 2.5. *Let β be a common divisor of r and some b_j. Then*

$$\sum_{i \equiv d \mod \beta} \sigma_i = 0 \quad \text{for any } d \mod \beta. \tag{2.15}$$

In other words, the average of σ_i over any coset of $\beta\mathbb{Z}/r \subset \mathbb{Z}/r$ is zero.

This just rewrites the zeros in the first line of (2.13). For example

$$\sigma_i(\tfrac{1}{14}(1,2,5,7)) = \frac{1}{14}(-2,-2,-1,\tfrac{1}{2},0,-\tfrac{1}{2},1,2,2,1,-\tfrac{1}{2},0,\tfrac{1}{2},-1), \tag{2.16}$$

with $\sigma_i + \sigma_{7+i} = \sum_{l=0}^{6} \sigma_{2l+i} = 0$ for each i.

The next result was first stated and proved in Buckley [5], Theorem 2.2, following the ideas of [YPG].

Theorem 2.6. *Let A and F be as in (2.10) and $\sigma_i = \sigma_i(\frac{1}{r}(b_1, \dots, b_n))$ as in (2.13). Then the polynomial $B(t) = A(t) \sum_{i=1}^{r} \sigma_{r-i} t^i$ is congruent to 1 modulo F. Equivalently,*

$$\operatorname{InvMod}(A, F, \gamma) \equiv \sum_{i=1}^{r} \sigma_{r-i} t^i \in \mathbb{Q}[t]/(F) \quad \text{for any } \gamma. \tag{2.17}$$

Proof. Substitute $t = \varepsilon$ any root of F in B and use (2.13) with the inverse value of ε. This gives

$$B(\varepsilon) = A(\varepsilon)\sum_{i=1}^{r}\sigma_{r-i}\varepsilon^i = \frac{A(\varepsilon)}{\prod_j(1-\varepsilon^{b_j})} = 1. \tag{2.18}$$

This holds for every root ε of F, so $B(t)-1$ is divisible by F, that is,

$$A(t)\sum_{i=1}^{r}\sigma_{r-i}t^i \equiv 1 \mod F. \quad \square \tag{2.19}$$

Algorithm 2.7. The Dedekind sums σ_i are obtained as the coefficients of an Inverse Mod polynomial. The point is just to average out the σ_i so that any coset mod β adds to 0, as in Lemma 2.5.

In the coprime case, first calculate $\mathrm{InvMod}(A,F,0) = \sum_{i=0}^{r-2} s_{r-i}t^i$ by the Euclidean algorithm. Then subtract aF where $a = \frac{1}{r}\sum_{i=0}^{r-2} s_{r-i}$, to give $\mathrm{InvMod}(A,F,0) - aF = \sum_{i=0}^{r-1}\sigma_{r-i}t^i$, with $\sigma_1 = -a$.

We use the following result in Section 6.

Corollary 2.8. *Assume all the b_i are coprime to r, so that $F = \frac{1-t^r}{1-t}$ and $d = \deg F = r-1$. Then for any γ,*

$$\begin{aligned}(1-t)^n\sum_{i=0}^{r-1}\sigma_{r-i}t^i &\equiv \mathrm{InvMod}\Big(\prod_{j=1}^{n}\frac{1-t^{b_j}}{1-t}, F\Big) \\ &\equiv \mathrm{InvMod}\Big(\frac{A}{(1-t)^n}, F, \gamma+1\Big) = \sum_{l=\gamma+1}^{\gamma+r-1}\theta_l t^l,\end{aligned} \tag{2.20}$$

with integral coefficients $\theta_l = \sum_{s=0}^{n}(-1)^s\binom{n}{s}(\sigma_{s-l}-\sigma_{s-\gamma}) \in \mathbb{Z}$.

Proof of the corollary. The inverse of $A/(1-t)^n$ modulo F is congruent to $(1-t)^n$ times $\mathrm{InvMod}(A,F,\gamma)$, that is, by Theorem 2.6, to

$$(1-t)^n\sum_{i=0}^{r-1}\sigma_{r-i}t^i = \sum_{s=0}^{n}\sum_{i=0}^{r-1}(-1)^s\binom{n}{s}\sigma_{r-i}t^{s+i}. \tag{2.21}$$

Now work modulo F, and shift the terms of the double sum into the range $[\gamma+1,\dots,\gamma+r-1]$. For terms with $s+i \not\equiv \gamma \mod r$, just shift i modulo r, bearing in mind that the Dedekind sum σ_{r-i} depends only on the subscript $r-i$ modulo r, and that $t^r \equiv 1$. For l in the range, putting together the terms with $s+i \equiv l \mod r$ gives $\sum_{s=0}^{n}(-1)^s\binom{n}{s}\sigma_{s-l}t^l$.

This leaves the terms with $s+i \equiv \gamma \mod r$; we handle these by subtracting off a multiple of $t^\gamma F = t^\gamma + t^{\gamma+1} + \cdots + t^{\gamma+r-1}$. This subtracts $\sum_{s=0}^{n}(-1)^s\binom{n}{s}\sigma_{s-\gamma}$ from every term in the range. This proves the equality in (2.20).

We now prove that $\theta_l \in \mathbb{Z}$. Plugging in the values of the Dedekind sums σ_i and summing over s by the binomial theorem gives

$$\begin{aligned} r\theta_l &= \sum_{s=0}^{n}(-1)^s\binom{n}{s}\sum_{\varepsilon\in\mu_r-\{1\}}\frac{\varepsilon^{s-l}-\varepsilon^{s-\gamma}}{(1-\varepsilon^{b_1})\cdots(1-\varepsilon^{b_n})} \\ &= \sum_{\varepsilon\in\mu_r-\{1\}}\frac{(\varepsilon^{r-l}-\varepsilon^{r-\gamma})(1-\varepsilon)^n}{(1-\varepsilon^{b_1})\cdots(1-\varepsilon^{b_n})} \\ &= \sum_{\varepsilon\in\mu_r-\{1\}}(\varepsilon^{r-l}-\varepsilon^{r-\gamma})\prod_{j=1}^{n}\frac{1-\varepsilon}{1-\varepsilon^{b_j}}. \end{aligned} \tag{2.22}$$

The second factor in the summand is a polynomial in ε with integral coefficients; indeed, the b_j are coprime to r, so

$$\frac{1-\varepsilon}{1-\varepsilon^{b_j}} = \frac{1-\varepsilon^{a_jb_j}}{1-\varepsilon^{b_j}} = 1+\varepsilon^{b_j}+\cdots+\varepsilon^{(a_j-1)b_j} \tag{2.23}$$

where $a_jb_j \equiv 1 \mod r$. Now $\sum_{\varepsilon\in\mu_r-\{1\}}\varepsilon^\beta \equiv -1 \mod r$ for every $\beta\in\mathbb{Z}$ (either $r-1$ if $\beta\equiv 0$ or -1 if $\beta\not\equiv 0$). Hence the expression in (2.22) is $\equiv 0 \mod r$, so that $\theta_l\in\mathbb{Z}$.

Better proof. Replace the InvMod of a product in (2.20) by the product of InvMods. Now $\operatorname{InvMod}\left(\frac{1-t^{b_j}}{1-t}, F, 1\right)$ is an integral polynomial; indeed, it is the ice cream function for $\frac{a_j}{r}$ where $a_j = \operatorname{InvMod}(b_j, r)$, by the calculation of 2.1, (2.5). □

Exercise 2.9. The ice cream function of 2.1 corresponds to $\frac{1}{7}(5)$: the periodic rounding loss of (2.9) is

$$\begin{aligned} \sum_{i=0}^{6}\sigma_{7-i}t^i &= \tfrac{1}{7}(3-3t^2+t^3-2t^4+2t^5-t^6) \\ &\equiv \operatorname{InvMod}\left(1-t^5, \tfrac{1-t^7}{1-t}\right), \end{aligned}$$

whereas

$$\begin{aligned} &(1-t)\times\tfrac{1}{7}(3-3t^2+t^3-2t^4+2t^5-t^6) \\ &\qquad\equiv t^3+t^5+t^7 = \operatorname{InvMod}\left(\frac{1-t^5}{1-t}, \frac{1-t^7}{1-t}, 1\right). \end{aligned}$$

Exercise 2.10 (Serre duality and Gorenstein symmetry)**.**

(1) Prove that X projectively Gorenstein of canonical weight k_X implies that $k_X + \sum_{j=1}^n b_j \equiv 0 \mod r$ for each $Q = \frac{1}{r}(b_1,\dots,b_n)$.
(2) Prove that the σ_i are $(-1)^n$ symmetric under $i\mapsto\sum b_j - i$. [Hint: replace $\varepsilon\mapsto\varepsilon^{-1}$ in the characterization (2.13) of σ_i, or in the formula (2.14).]
(3) Now suppose coprime, let θ_l be as in Corollary 2.8, and let k_X be the canonical weight. Prove that $l_1+l_2\equiv k+n \mod r$ implies $\theta_{l_1}=\theta_{l_2}$.

Example 2.11. The weighted projective line $X = \mathbb{P}(5,7)$ has $k_X = -12$, and has two orbifold points of type $\frac{1}{7}(5)$ and $\frac{1}{5}(2)$. Its Hilbert series

$$P_X(t) = \frac{1}{(1-t^5)(1-t^7)} \tag{2.24}$$

satisfies Theorem 1.1: since $c = -10 < 0$, the initial term $P_I = 0$. Then

$$\begin{aligned} P_X(t) &= P_{\text{orb}}\big(\tfrac{1}{7}(5), -12\big) + P_{\text{orb}}\big(\tfrac{1}{5}(2), -12\big) \\ &= \frac{t^{-4}+t^{-2}+1}{(1-t)(1-t^7)} + \frac{-t^{-4}-t^{-2}}{(1-t)(1-t^5)}, \end{aligned}$$

where $-t^{-4} - t^{-2} = \text{InvMod}(\frac{1-t^2}{1-t}, \frac{1-t^5}{1-t}, -4)$.

Exercise 2.12. Games with the ice cream function of 2.1:

(1) An elliptic curve polarized by $A = \frac{3}{7}P$ embeds as $C_{15} \subset \mathbb{P}(1,5,7)$ with canonical weight 2, that is $K_{C,\text{orb}} = 2A = \frac{6}{7}P$.
(2) A curve of genus 2 polarized by $P + \frac{3}{7}Q$ with P a Weierstrass point embeds in $\mathbb{P}(1,2,3,5,7)$ as a Pfaffian with Hilbert numerator
$$1 - t^6 - t^7 - t^8 - t^9 - t^{10} + t^{10} + t^{11} + t^{12} + t^{13} + t^{14} - t^{20}.$$
(3) $\mathbb{P}^1$ polarized by $-P + \sum_3 \frac{3}{7}Q_i$ (that is, three 1/7 orbifold points) embeds in codimension 4 in $\mathbb{P}(3,5,5,7,7,7)$ with Hilbert numerator
$$1 - 3t^{10} - 3t^{12} - 3t^{14} + 2t^{15} + 6t^{17} + 6t^{19} + 2t^{21} - 3t^{22} - 3t^{24} - 3t^{26} + t^{36}.$$

```
Ice Cream

Dedekind sums are the same as InvMod plus a bit of averaging
to zero. They give the R periodicity, with denom (1-t^R). We
have a mechanism, at least in the isolated case, that
replaces the fractional term based on Dedekind sums with an
integral symmetric term with denominator (1-t)^n*(1-t^r).
The trick in the isolated case is not to take InvMod(A,F),
but instead InvMod(A/(1-t)^n,F).

The whole game is
  "taking Inverse of product (1-bi) modulo (1-t^r)".   (*)
of course this is nonsense if we don't take account of the
common factors between (1-t^bi) and (1-t^r). The Ice Cream
solution to this problem is to take out *all* the common
factors from both.

If we are doing 1/r(b1,.. bn), set ri = hcf(bi, r), so that
1-t^ri is the hcf of 1-t^bi and 1-t^r, and mu_ri fixes the
xi axis pointwise. The denominator of the Ice Cream function
is
  D = product (1-t^ri) * (1-t^r).
```

```
For the numerator, take
  A = product (1-t^bi) / (1-t^ri)
and set
  F = (1-t^r) / hcf(1-t^r, prod(1-t^ri)
We then replace the original motivating idea (*) by taking
InverseMod(A,F). The remaining issue is to shift the support
into a suitable range so that Num/D is Gorenstein symmetric
of degree k.

Usually the denominator D has degree quite a lot bigger than
n+r. The length of the range should be Degree(F) (usually
quite a lot smaller than r); the Numerator must be
Gorenstein symmetric of degree Degree(D)+k, so its range
should be centred at (Degree(D)+k)/2, and so it should start
at (Degree(D)-Degree(F)+k)/2.

e.g. 1/14[1,2,5,7]. The ri are [1,2,1,7]. The A is
(1-t^5)/(1-t). The F = 1-t+t^2-t^3+t^4-t^5+t^6 (the
cyclotomic polynomial of order 14), degree 6. The
denominator D is (1-t)^2*(1-t^2)*(1-t^7)*(1-t^14) has degree
25. k = -1, so the numerator should be Gorenstein symmetric
of degree 24, centred at 12, so supported in [10,14].

25 - 6 - 1 = 18, so half of it is 9

/* I think something like this works. */

function QQorb(r,LL,k)
 L := [Integers() | i : i in LL]; // this allows empty list
 if (k + &+L) mod r ne 0
    then error "Error: Canonical weight not compatible";
 end if;
n := #L;
rr := [GCD(l,r) : l in L];
A := &*[(1-t^(L[i])) div (1-t^(rr[i])) : i in [1..#L]];
F := (1-t^r) div GCD(1-t^r, &*[1-t^l : l in rr]);
D := Denom(rr cat [r]);
l := Floor((Degree(D)-Degree(F)+k+1)/2);
  // Kludge avoids programming Laurent polynomials if l < 0.
 de := Maximum(0,Ceiling(-l/r));
 m := l + de*r;
 G, al, be_throwaway := XGCD(t^m*A,F);
 return t^m*al/(D*t^(de*r));
end function;
```

3. K3 surfaces and Fano 3-folds

Our Main Theorem 1.1 correlates simply with known results on K3s and Fano 3-folds (see Altınok, Brown and Reid [2]). Let (S, D) be a polarized K3 surface with a basket of orbifold points $\mathcal{B} = \{\frac{1}{r}(a, r-a)\}$. [YPG], Appendix to Section 8, gives

$$\sigma_i = \frac{r^2-1}{12r} - \frac{\overline{bi}(r-\overline{bi})}{2r}, \tag{3.1}$$

where $ab = 1$ modulo r and $\overline{}$ denotes the smallest nonnegative residue mod r. By Theorem 2.6,

$$\mathrm{InvMod}\Big((1-t^a)(1-t^{r-a}), \frac{1-t^r}{1-t}\Big) = -\frac{1}{2r}\sum_{i=1}^{r-1} \overline{bi}(r-\overline{bi})t^i$$

and

$$\mathrm{InvMod}\Big(\frac{(1-t^a)(1-t^{r-a})}{(1-t)^2}, \frac{1-t^r}{1-t}\Big) \equiv -\frac{(1-t)^2}{2r}\sum_{i=1}^{r-1} \overline{bi}(r-\overline{bi})t^i$$

An immediate application of RR for surfaces [2, Theorem 4.6] gives that the Hilbert series is

$$P_S(t) = \frac{1+t}{1-t} + \frac{t+t^2}{(1-t)^3}\cdot\frac{D^2}{2} - \sum_{\mathcal{B}} \frac{1}{1-t^r}\sum_{i=1}^{r-1}\frac{\overline{bi}(r-\overline{bi})}{2r}t^i. \tag{3.2}$$

Comparing the coefficients of t in (3.2) yields

$$D^2 = 2g - 2 + \sum_{\mathcal{B}} \frac{b(r-b)}{r}, \tag{3.3}$$

where the genus g is defined by $P_1 = h^0(S, \mathcal{O}_S(D)) = g+1$ Then $P_S(t) = P_I + \sum_{\mathcal{B}} P_{\mathrm{orb}}$, where

$$P_I = \frac{1+(g-2)t+(g-2)t^2+t^3}{(1-t)^3} = \frac{1+t}{1-t} + (g-1)\frac{t+t^2}{(1-t)^3}, \tag{3.4}$$

and

$$P_{\mathrm{orb}} = \frac{\mathrm{InvMod}\Big(\frac{(1-t^b)(1-t^{r-b})}{(1-t)^2}, \frac{1-t^r}{1-t}, 2\Big)}{(1-t)^2(1-t^r)}. \tag{3.5}$$

One checks that $P_{\mathrm{orb}} = \frac{t+t^2}{(1-t)^3}\cdot\frac{b(r-b)}{2r} - \frac{1}{1-t^r}\sum_{i=1}^{r-1}\frac{\overline{bi}(r-\overline{bi})}{2r}t^i$, as above.

Corollary 3.1. *Let V be a $\mathbb{Q}$-Fano 3-fold with basket $\mathcal{B} = \big\{\frac{1}{r}(1, b, r-b)\big\}$ of terminal quotient singularities. Its anticanonical ring has Hilbert series of the form $P_V(t) = P_I + \sum_{\mathcal{B}} P_{\mathrm{orb}}$, with*

$$P_I = \frac{1+(g-2)t+(g-2)t^2+t^3}{(1-t)^4} \tag{3.6}$$

where $h^0(-K_X) = g + 2$ *and* $-K^3 = 2g - 2 + \sum \frac{b(r-b)}{r}$*, and*

$$P_{\mathrm{orb}} = \frac{\mathrm{InvMod}\Big(\frac{(1-t)(1-t^b)(1-t^{r-b})}{(1-t)^3}, \frac{1-t^r}{1-t}, 2\Big)}{(1-t)^3(1-t^r)}. \tag{3.7}$$

Proof. By [2, Theorem 4.6] the Hilbert series of $(V, -K_V)$ equals

$$P_V(t) = \frac{1+t}{(1-t)^2} - \frac{t+t^2}{(1-t)^4} \cdot \frac{K_V^3}{2} - \sum_{\mathcal{B}} \frac{1}{(1-t)(1-t^r)} \sum_{i=1}^{r-1} \frac{\overline{bi}(r-\overline{bi})}{2r} t^i. \quad \square \tag{3.8}$$

Example 3.2. Consider surfaces

- $S_5 \subset \mathbb{P}(1,1,1,2)$ with an orbifold point of type $\frac{1}{2}(1,1)$ at $Q = (0,0,0,1)$,
- $S_7 \subset \mathbb{P}(1,1,2,3)$ with orbifold points of types $\frac{1}{2}(1,1)$ and ${}_2\big(\frac{1}{3}(1,2)\big)$,
- $S_{11} \subset \mathbb{P}(1,2,3,5)$ with orbifold points of type $\frac{1}{2}(1,1)$, $\frac{1}{3}(1,2)$, $\frac{1}{5}(2,3)$,

where S_i is a general surface of degree i, for $i = 5, 7, 11$ in the corresponding weighted projective space. All three surfaces have $k_{S_i} = 0$ and $c = 3$. The Hilbert series parses as $P_{S_i}(t) = P_I + \sum_{\mathcal{B}_i} P_{\mathrm{orb}}$

$$\begin{aligned}
P_{S_5}(t) &= \frac{1-t^5}{(1-t)^3(1-t^2)} \\
&= \frac{1+t^3}{(1-t)^3} + \frac{t^2}{(1-t)^2(1-t^2)}, \\
P_{S_7}(t) &= \frac{1-t^7}{(1-t)^2(1-t^2)(1-t^3)} \\
&= \frac{1-t-t^2+t^3}{(1-t)^3} + \frac{t^2}{(1-t)^2(1-t^2)} + \frac{t^2+t^3}{(1-t)^2(1-t^3)}, \\
P_{S_{11}}(t) &= \frac{1-t^{11}}{(1-t)^2(1-t^2)(1-t^3)} = \frac{1-2t-2t^2+t^3}{(1-t)^3} \\
&\quad + \frac{t^2}{(1-t)^2(1-t^2)} + \frac{t^2+t^3}{(1-t)^2(1-t^3)} + \frac{2t^2+t^3+t^4+2t^5}{(1-t)^2(1-t^5)}.
\end{aligned}$$

4. Calabi–Yau 3-folds

Let (X, D) be a Calabi–Yau threefold polarized by a $\mathbb{Q}$-Cartier Weil divisor D. Assume that (X, D) is quasismooth and well formed with the following orbifold locus:

- isolated points Q of type ${}_{s-1}\big(\frac{1}{s}(a_1, a_2, a_3)\big)$, such that $a_1 + a_2 + a_3 = 0 \mod s$
- curves C of generic type ${}_k\big(\frac{1}{r}(1, -1)\big)$.

Then by [6, Corollary 3.3.], the Hilbert series can be written in the compact form

$$P_X(t) = 1 + \frac{D^3}{6} \cdot \frac{t^3+4t^2+t}{(1-t)^4} + \frac{c_2(X)\cdot D}{12} \cdot \frac{t}{(1-t)^2} + \sum_Q \tilde{P}_Q(t) + \sum_C \tilde{P}_C(t),$$

where every curve C contributes

$$\tilde{P}_C(t) = -\deg D|_C \left(\frac{1}{1-t^r} \sum_{i=1}^{r-1} i \frac{\overline{ik}(r-\overline{ik})}{2r} t^i + \frac{rt^r}{(1-t^r)^2} \sum_{i=1}^{r-1} \frac{\overline{ik}(r-\overline{ik})}{2r} t^i \right) + \frac{N_C}{12r^2} \cdot \frac{1}{1-t^r} \sum_{i=1}^{r-1} \overline{ik}(r-\overline{ik})(r-2\cdot\overline{ik}) t^i$$

and every singular point Q contributes

$$\tilde{P}_Q(t) = \frac{1}{1-t^s} \sum_{i=1}^{s-1} c_Q(iD) t^i, \text{ with}$$

$$c_Q(iD) = \frac{1}{s} \sum_{\varepsilon \in \mu_s} \frac{\varepsilon^{-i}-1}{(1-\varepsilon^{-a_1})(1-\varepsilon^{-a_2})(1-\varepsilon^{-a_3})}.$$

Theorem 4.1. *Let (X, D) be a polarized Calabi–Yau 3-fold, and assume that its singular locus $\mathcal{B} = \{Q, C\}$ consists of isolated points and Du Val curves. Then the Hilbert series of X is of the form*

$$P_X(t) = P_I + \sum_Q P_{\text{orb}} + \sum_C P_{\text{per}} + \sum_C P_{\text{grow}}, \tag{4.1}$$

where

$$P_{\text{grow}} = \deg D|_C \cdot \frac{rt^r}{(1-t)^2(1-t^r)^2} \cdot \left(\textit{inverse of } \frac{(1-t^a)(1-t^{r-a})}{(1-t)^2} \textit{ modulo } \frac{1-t^r}{1-t} \textit{ supported in } \left[-r + \left\lceil \frac{r}{2} \right\rceil + 2, \dots, \left\lfloor \frac{r}{2} \right\rfloor \right] \right),$$

$$P_{\text{per}} = \frac{1}{(1-t)^3(1-t^r)} \cdot \left(\deg D|_C \cdot D_C\textit{-Num} - \frac{N_C}{2r} \cdot N_C\textit{-Num} \right)$$

and D_C-Num, N_C-Num *are Gorenstein symmetric polynomials with support* $[3, \dots, r]$.

Proof. For clarity we restrict to X having one isolated orbifold point Q of type ${}_{s-1}\big(\frac{1}{s}(a_1, a_2, a_3)\big)$ and one orbifold curve C of type ${}_k\big(\frac{1}{r}(1, -1)\big)$. For a bigger basket $\mathcal{B}$, we just write $\sum_Q$, $\sum_C$ in front of contributions.

Write $P_i = h^0(X, \mathcal{O}_X(iD))$ for the ith plurigenus of X. Comparing the coefficients of t and t^2 in the Hilbert series yields

$$\begin{aligned} D^3 &= P_2 - 2P_1 + 2c_Q(D) - c_Q(2D) \\ &\quad + \frac{1}{r}\deg D|_C\left(-k(r-k) + \overline{2k}(r-\overline{2k})\right) \\ &\quad + \frac{N_C}{12r^2}\Big(2k(r-k)(r-2k) - \overline{2k}(r-\overline{2k})(r-2\cdot\overline{2k})\Big), \\ c_2(X)\cdot D &= -2\Big[8c_Q(D) - c_Q(2D) - 8P_1 + P_2 \\ &\quad + \frac{1}{r}\deg D|_C\left(-4k(r-k) + \overline{2k}(r-\overline{2k})\right) \\ &\quad + \frac{N_C}{12r^2}\Big(8k(r-k)(r-2k) - \overline{2k}(r-\overline{2k})(r-2\cdot\overline{2k})\Big)\Big]. \end{aligned}$$

Then the Hilbert series is

$$\begin{aligned} P_X(t) &= \frac{1 + (P_1-4)t + (P_2 - 4P_1 + 6)t^2 + (P_1-4)t^3 + t^4}{(1-t)^4} \\ &\quad - \frac{t^2c_Q(2D) + (t - 4t^2 + t^3)c_Q(D)}{(1-t)^4} + \tilde{P}_Q(t) \\ &\quad - \deg D|_C\left(\frac{1}{1-t^r}\sum_{i=1}^{r-1} i\frac{\overline{ik}(r-\overline{ik})}{2r}t^i + \frac{rt^r}{(1-t^r)^2}\sum_{i=1}^{r-1}\frac{\overline{ik}(r-\overline{ik})}{2r}t^i\right) \\ &\quad + \frac{N_C}{12r^2}\cdot\frac{1}{1-t^r}\sum_{i=1}^{r-1}\overline{ik}(r-\overline{ik})(r-2\cdot\overline{ik})t^i \\ &\quad + \frac{1}{2r}\deg D|_C\left(\frac{t-4t^2+t^3}{(1-t)^4}k(r-k) + \frac{2t^2}{(1-t)^4}\overline{2k}(r-\overline{2k})\right) \\ &\quad - \frac{N_C}{12r^2}\left(\frac{t-4t^2+t^3}{(1-t)^4}k(r-k)(r-2k) + \frac{t^2}{(1-t)^4}\overline{2k}(r-\overline{2k})(r-2\cdot\overline{2k})\right). \end{aligned}$$

First observe that the first part is $P_I(t)$: it equals $P_X(t)$ up to degree $\lfloor\frac{4}{2}\rfloor = 2$ and its numerator is Gorenstein symmetric of degree $c = 4$.

Next we prove that

$$-\frac{t^2c_Q(2D) + (t-4t^2+t^3)c_Q(D)}{(1-t)^4} + \tilde{P}_Q(t) = P_{\mathrm{orb}}. \tag{4.2}$$

In Dedekind sum notation, we get

(4.3)
$$\frac{-\Big(t^2(\sigma_{-2}-\sigma_0) + (t-4t^2+t^3)(\sigma_{-1}-\sigma_0)\Big)\sum_{i=0}^{s-1} t^i + (1-t)^3\sum_{i=0}^{s-1}(\sigma_{-i}-\sigma_0)t^i}{(1-t)^3(1-t^s)}.$$

By Theorem 2.6, the numerator equals the inverse of $\frac{(1-t^{a_1})(1-t^{a_2})(1-t^{a_3})}{(1-t)^3}$ modulo $\frac{1-t^s}{1-t}$. It is also easy to check that the numerator has the coefficients at $1, t, t^2, t^{s+1}$ and t^{s+2} all 0, since $a_1+a_2+a_3 = 0 \mod s$ implies $\sigma_{-i} = -\sigma_i$. Thus it is supported in $[3, 4, \dots, s]$.

Clearly P_I and P_{orb} are invariant under $t \mapsto \frac{1}{t}$. This is compatible with Serre duality, which for an n-fold X implies the functional equation $(-1)^{n+1}P_X(t) = t^{k_X}P_X(1/t)$ (compare Proposition 6.4 and Exercise 2.10).

It remains to analyse the curve contribution. We parse it into three sections that are also preserved under $t \mapsto \frac{1}{t}$. The terms involving N_C sum into

$$-\frac{N_C}{2r} \cdot \frac{N_C\text{-Num}}{(1-t)^3(1-t^r)}. \tag{4.4}$$

One checks directly that N_C-Num has zero coefficients at $1, t, t^2, t^{r+1}, t^{r+2}$ and is thus supported in $[3, \dots, r]$. The following exercise will be useful.

Exercise 4.2. For all integers i and $ak = 1 \mod r$, prove the equality:

$$\begin{aligned} -\frac{1}{2}\sigma_i(\tfrac{1}{r}(a, r-a)) + \tfrac{1}{2}\sigma_0(\tfrac{1}{r}(a, r-a)) \\ + \sigma_i(\tfrac{1}{r}(a, r-a, r-a)) - \sigma_0(\tfrac{1}{r}(a, r-a, r-a)) \\ = \frac{1}{12r}\overline{ki}(r-\overline{ki})(r-2\cdot\overline{ki}). \end{aligned}$$

By the above exercise N_C-Num equals

$$-2(1-t)^3\sum_{i=1}^{r-1}\Big(\sigma_i\big(\tfrac{1}{r}(a, r-a, r-a)\big) - \tfrac{1}{2}\sigma_i\big(\frac{1}{r}(a, r-a)\big)\Big)t^i. \tag{4.5}$$

The definition of σ_i gives

$$\begin{aligned} N_C\text{-Num} &\equiv -(1-t)^3\sum_{i=1}^{r-1}\Big(\sigma_i\big(\tfrac{1}{r}(a, r-a, r-a)\big) \\ &\qquad + \sigma_{i+r-a}\big(\tfrac{1}{r}(a, r-a, r-a)\big)\Big)t^i \\ &\equiv -(1-t)^3(1+t^{r-a})\sum_{i=1}^{r-1}\sigma_i\big(\tfrac{1}{r}(a, r-a, r-a)\big)t^i \\ &\equiv (1+t^{r-a})(1-t)^3\sum_{i=1}^{r-1}\sigma_{r-i}\big(\tfrac{1}{r}(a, r-a, r-a)\big)t^i \mod \frac{1-t^r}{1-t}. \end{aligned}$$

This, together with Theorem 2.6 shows, that N_C-Num equals $(1+t^{r-a})$ times the inverse of $\frac{(1-t^a)(1-t^{r-a})^2}{(1-t)^3}$ modulo $\frac{1-t^r}{1-t}$. Thus N_C-Num can be obtained as an ice cream function. In other words,

$$N_C\text{-Num} \times \frac{(1-t^a)(1-t^{r-a})^2}{(1-t)^3} \equiv 1 + t^{r-a} \mod \frac{1-t^r}{1-t}. \tag{4.6}$$

On the other hand, the terms involving $\deg D|_C$ sum to

$$\deg D|_C \cdot \frac{rt^r}{(1-t)^2(1-t^r)^2} \cdot \Big(\text{inverse of } \frac{(1-t^a)(1-t^{r-a})}{(1-t)^2} \text{ mod } \frac{1-t^r}{1-t} \\ \text{supported in } \Big[-r+\Big\lceil\frac{r}{2}\Big\rceil+2,\dots,\Big\lfloor\frac{r}{2}\Big\rfloor\Big]\Big) + \deg D|_C \cdot \frac{D_C\text{-Num}}{(1-t)^3(1-t^r)},$$

where D_C-Num denotes the polynomial

$$\sum_{i=3}^{\lceil \frac{r}{2}\rceil+1} \Big(i\sigma_i - 3(i-1)\sigma_{i-1} + 3(i-2)\sigma_{i-2} - (i-3)\sigma_{i-3} + 2(\sigma_1-\sigma_2)\Big)t^i \\ + \sum_{i=\lceil \frac{r}{2}\rceil+2}^{r} \Big((i-r)\sigma_i - 3(i-1-r)\sigma_{i-1} \\ + 3(i-2-r)\sigma_{i-2} - (i-3-r)\sigma_{i-3} + 2(\sigma_1-\sigma_2)\Big)t^i.$$

Note that the terms at t^{3+i} and t^{r-i} in D_C-Num are the same, which implies Gorenstein symmetry. The above result is a long, but elementary exercise using only Corollary 2.8 for $\frac{(1-t^a)(1-t^{r-a})}{(1-t)^2}$ and

$$\sigma_i\big(\tfrac{1}{r}(a,r-a)\big) = \frac{r^2-1}{12r} - \frac{\overline{ki}(r-\overline{ki})}{2r}, \tag{4.7}$$

where $ak = 1 \mod r$. □

Lemma 4.3. *D_C-Num can be evaluated from an ice cream function.*

Proof. Consider $\mathrm{Inv}(t)$:= the inverse of $(1-t^a)(1-t^{r-a})$ modulo $\frac{1-t^r}{1-t}$ supported in $[1,\dots,r-1]$. Then $\mathrm{Inv}(t) = \sum_{i=1}^{r-1}(\sigma_i-\sigma_0)t^i$ and

$$(1-t)^3\cdot \mathrm{Inv}(t)' = \sigma_1-\sigma_0 + (2\sigma_2 - 3\sigma_1+\sigma_0)t \\ + \sum_{i=3}^{r}\left(i\sigma_i - 3(i-1)\sigma_{i-1} + 3(i-2)\sigma_{i-2} - (i-3)\sigma_{i-3}\right)t^{i-1} \\ + \text{ terms at } t^r \text{ and } t^{r+1}.$$

The coefficients at $t^3, t^4, \dots, t^{\lceil \frac{r}{2}\rceil+1}$ in the polynomial

$$t\left((1-t)^3\cdot \mathrm{Inv}(t)' - (\sigma_1-\sigma_0)\frac{1-t^r}{1-t} - (2\sigma_2-3\sigma_1+\sigma_0)t\frac{1-t^r}{1-t}\right) \tag{4.8}$$

are exactly the left hand side of the symmetric polynomial D_C-Num. □

Example 4.4. Consider $X_{66} \subset \mathbb{P}(5,6,11,11,33)$. It is a Calabi–Yau with ${}_{-1}\big(\frac{1}{3}(2,2,2)\big)$ and ${}_{-1}\big(\frac{1}{5}(1,1,3)\big)$ points and ${}_{-1}\big(\frac{1}{11}(5,6)\big)$ curve of degree $\frac{2}{11}$.

The Hilbert series equals

$$(4.9)\quad \frac{1-t^{66}}{(1-t^5)(1-t^6)(1-t^{11})^2(1-t^{33})} =$$

$$= P_I + P_{\mathrm{orb}\,1/3} + P_{\mathrm{orb}\,1/5} + P_{\mathrm{grow}\,1/11} + P_{\mathrm{per}\,1/11}$$

$$= \frac{1-4t+6t^2-4t^3+t^4}{(1-t)^4} - \frac{t^3}{(1-t^3)(1-t)^3} + \frac{t^3(1+t^2)}{(1-t^5)(1-t)^3} + \frac{2t^8(1+t+t^2+t^3-t^4+t^5+t^6+t^7+t^8)}{(1-t)^2(1-t^{11})^2} - \frac{t^6+t^8}{(1-t)^3(1-t^{11})}.$$

Moreover,
(4.10)

$$\begin{aligned} -t^6-t^8 &= \frac{2}{11}\cdot D_C\text{-Num} - \frac{1}{11}\cdot N_C\text{-Num} \\ &= \frac{2}{11}(2t^2+2t^3+2t^4-4t^5-t^6-4t^7+2t^8+2t^9+2t^{10}) \\ &\quad - \frac{1}{11}(4t^3+4t^4+4t^5+3t^6-2t^7+3t^8+4t^9+4t^{10}+4t^{11}). \end{aligned}$$

Remark 4.5. We restricted our study to isolated singular points, since $P_{\mathrm{orb}}(t)$ for a dissident point $\frac{1}{s}(a_1,a_2,a_3)$ is not uniquely determined as an $\mathcal{I}nv\mathcal{M}od$ function supported on $[3,\dots,s+1]$. The main reason is that contributions from a dissident point on C already incorporate contributions from C.

For example, $X_{40} \subset \mathbb{P}(2,5,8,10,15)$ is a Calabi–Yau with a ${}_{-1}\big(\frac{1}{15}(2,5,8)\big)$ dissident point on a degree $\frac{4}{15}$ curve of generic type $\frac{1}{5}(2,3)$ and with a $\frac{1}{2}(1,1)$ orbifold line. In the proof of Theorem 4.1 we computed the Hilbert series

$$\begin{aligned} &\frac{1-t^{40}}{(1-t^2)(1-t^5)(1-t^8)(1-t^{10})(1-t^{15})} \\ &= P_I + P_{\mathrm{grow}\,1/2} + P_{\mathrm{per}\,1/2} + P_{\mathrm{orb}\,1/15} + P_{\mathrm{grow}\,1/5} + P_{\mathrm{per}\,1/5} \\ &= \frac{1-4t+7t^2-4t^3+t^4}{(1-t)^4} - \frac{t^3}{(1-t)^2(1-t^2)^2} + \\ &\quad \frac{-\frac{1}{3}t^3 * \frac{2}{3}t^4 - \frac{2}{3}t^5 * \frac{2}{3}t^6 - \frac{2}{3}t^7 * t^8 - \frac{4}{3}t^9 * t^{10} - \frac{2}{3}t^{11} * \frac{2}{3}t^{12} - \frac{2}{3}t^{13} * \frac{2}{3}t^{14} - \frac{1}{3}t^{15}}{(1-t)^3(1-t^{15})} \\ &\quad + \frac{4}{15}\frac{5t^5(1-t+t^2)}{(1-t)^2(1-t^5)^2} - \frac{4}{3}\frac{2t^3-t^4+2t^5}{(1-t)^3(1-t^5)}. \end{aligned}$$

Note that the only nonzero contribution from the $\frac{1}{2}$ curve is $P_{\text{grow}1/2} = \frac{1}{2}\frac{2(-t^3)}{(1-t)^2(1-t^2)^2}$. Note also, that the numerator of $P_{\text{per}1/5}$ equals

$$\begin{aligned}-\frac{4}{3}(2t^3 - t^4 + 2t^5) &= \frac{4}{15}\cdot D_C\text{-Num} - \frac{4}{5}\cdot N_C\text{-Num}\\ &= \frac{4}{15}(-t^3 - t^4 - t^5) - \frac{4}{5}(3t^3 - 2t^4 + 3t^5).\end{aligned}$$

On the other hand, $P_{\text{orb}1/15}$ is not unique modulo $1 + t^5 + t^{10} = \frac{1-t^{15}}{(1-t^5)}$. The computer output for the inverse of $\frac{(1-t^2)(1-t^5)(1-t^8)}{(1-t)^3}$ modulo $1+t^5+t^{10}$ supported in $[5,\dots,13]$ is

$$-\frac{1}{3}t^5 + \frac{2}{3}t^6 - \frac{2}{3}t^7 + \frac{4}{3}t^8 - 2t^9 + \frac{4}{3}t^{10} - \frac{2}{3}t^{11} + \frac{2}{3}t^{12} - \frac{1}{3}t^{13}, \tag{4.11}$$

which relates it to $P_{\text{orb}1/15}$ in the following way

$$\begin{aligned}P_{\text{orb}1/15} = &\frac{-\frac{1}{3}t^5 + \frac{2}{3}t^6 - \frac{2}{3}t^7 + \frac{4}{3}t^8 - 2t^9 + \frac{4}{3}t^{10} - \frac{2}{3}t^{11} + \frac{2}{3}t^{12} - \frac{1}{3}t^{13}}{(1-t)^3(1-t^{15})}\\ &+ \frac{-\frac{1}{3}t^3 + \frac{2}{3}t^4 - \frac{1}{3}t^5}{(1-t)^3(1-t^5)}.\end{aligned}$$

In a mysterious way, P_{per} and the contribution from P_{orb} to the curve always sum into a numerator with integer coefficients. In our example

$$-\frac{4}{3}(2t^3 - t^4 + 2t^5) + (-\frac{1}{3}t^3 + \frac{2}{3}t^4 - \frac{1}{3}t^5) = -3t^3 + 2t^4 - 3t^5. \tag{4.12}$$

It is work in progress to determine such contributions explicitly.

However, we are able to parse the Hilbert series for Calabi–Yaus with dissident points on $\frac{1}{2}(1,-1)$ curves.

Theorem 4.6. *The Hilbert series of a Calabi–Yau threefold X with orbifold locus*

- *points Q of type ${}_{s-1}\big(\frac{1}{s}(a_1,a_2,a_3)\big)$ with $a_1+a_2+a_3 = 0$ modulo s,*
- *curves C of generic type $\frac{1}{2}(1,-1)$ with index τ_C,*

is of the form

$$P_X(t) = P_I + \sum_Q P_{\text{orb}} - \sum_C \deg D|_C \frac{2t^3}{(1-t)^2(1-t^2)^2}. \tag{4.13}$$

Proof. We repeat the steps in the proof of Theorem 4.1, but also take into account the dissident points on C. For a dissident point $Q \in C$ we have $\text{hcf}(a_i, s) = 2$ for some $i = 1,2,3$ and therefore

$$\text{hcf}\left((1-t^{a_1})(1-t^{a_2})(1-t^{a_3}), \frac{1-t^s}{1-t}\right) = 1 + t. \tag{4.14}$$

Note that most of the contributions due to the curves are 0. We get

$$P_X(t) = P_I + \sum_Q P_{\mathrm{orb}} - \sum_C \deg D|_C \frac{2t^3}{(1-t)^2(1-t^2)^2}, \tag{4.15}$$

where P_I equals $P_X(t)$ up to degree 2 and has a Gorenstein symmetric numerator of degree 4 and

(4.16)

$$P_{\mathrm{orb}} = \frac{\text{inverse of } \frac{(1-t^{a_1})(1-t^{a_2})(1-t^{a_3})}{(1-t)^3} \bmod \frac{1-t^s}{1-t} \text{ supported in } [3,\dots,s]}{(1-t)^3(1-t^s)}. \qquad \square$$

Examples 4.7. We test Theorem 4.6 on three Calabi–Yau hypersurfaces.

(1) $X_{10} \subset \mathbb{P}(11224)$ is a Calabi–Yau with a $\frac{1}{2}(1,1)$ curve of degree $\frac{10}{8}$ passing through a dissident point of type ${}_{-1}\big(\frac{1}{4}(1,1,2)\big)$. We used the computer code in Section 7 to verify

$$\begin{aligned} P_X(t) &= \frac{1-t^{10}}{(1-t)^2(1-t^2)^2(1-t^4)} \\ &= \frac{1-2t+3t^2-2t^3+t^4}{(1-t)^4} + \frac{t^3+t^4}{2(1-t)^3(1-t^4)} - \frac{10}{8}\cdot\frac{2t^3}{(1-t)^2(1-t^2)^2}. \end{aligned}$$

(2) $X_{11} \subset \mathbb{P}(11225)$ is a Calabi–Yau with a $\frac{1}{2}(1,1)$ curve of deg $\frac{1}{2}$ and an isolated point of type ${}_{-1}\big(\frac{1}{5}(1,2,2)\big)$. Check that

$$\begin{aligned} P_X(t) &= \frac{1-t^{11}}{(1-t)^2(1-t^2)^2(1-t^5)} \\ &= \frac{1-2t+3t^2-2t^3+t^4}{(1-t)^4} + \frac{-t^3+t^4-t^5}{(1-t)^3(1-t^5)} - \frac{1}{2}\cdot\frac{2t^3}{(1-t)^2(1-t^2)^2}. \end{aligned}$$

(3) $X_{18} \subset \mathbb{P}(1,1,2,6,8)$ is a Calabi–Yau with a $\frac{1}{2}(1,1)$ curve of deg $\frac{18}{6\cdot 8}$ and a dissident point of type ${}_{-1}\big(\frac{1}{8}(1,1,6)\big)$. Then

$$\begin{aligned} P_X(t) &= \frac{1-t^{18}}{(1-t)^2(1-t^2)(1-t^6)(1-t^8)} \\ &= \frac{1-2t+2t^2-2t^3+t^4}{(1-t)^4} + \frac{3t^3+t^4+2t^5+2t^6+t^7+3t^8}{4(1-t)^3(1-t^8)} \\ &\quad - \frac{3}{8}\cdot\frac{2t^3}{(1-t)^2(1-t^2)^2}. \end{aligned}$$

5. The Riemann–Roch formula

In this section we show that formulas of type (1.1) for polarized orbifolds exist and how to interpret them. This was done first in [YPG] for isolated points and in [6] for orbifold curves of type $\frac{1}{r}(b, r-b)$.

5.1. **The existence of the Riemann–Roch formula.** Let (X, D) be a normal projective n-fold which is quasismooth and well formed (with orbifold locus of any dimension). Choose a projective resolution of the quotient singularities $f\colon Y \to X$. For the divisorial sheaf $\mathcal{O}_X(D)$, define $\mathcal{L} = f^*(\mathcal{O}_X(D))/(\text{torsion})$. Then $\mathcal{L}$ is a rank 1 torsion free sheaf on Y.

For an ample Cartier divisor H on X and some integers m, N, there exists a surjection $\mathcal{O}_X^N(-mH) \twoheadrightarrow \mathcal{O}_X(D)$ on X which pulls back to a surjection $\mathcal{O}_Y^N \otimes f^*\mathcal{O}_X(-mH) \twoheadrightarrow \mathcal{L}$ on Y. This yields an exact sequence

$$0 \to \mathcal{K} \to \mathcal{O}_Y^N \otimes f^*\mathcal{O}_X(-mH) \to \mathcal{L} \to 0. \tag{5.1}$$

Now $f_*\mathcal{O}_Y = \mathcal{O}_X$ since X is normal and $R^i f_*\mathcal{O}_Y = 0$ for $i > 0$ since quotient singularities are rational in characteristic zero. Applying f_* to (5.1) gives

$$\begin{array}{cccccccc} 0 \to & f_*\mathcal{K} & \to & \mathcal{O}_X^N(-mH) & \to & f_*\mathcal{L} & \to & \\ & R^1 f_*\mathcal{K} & \to & 0 & \to & R^1 f_*\mathcal{L} & \to & \\ & \vdots & \to & 0 & \to & \vdots & & \\ & R^{n-1} f_*\mathcal{K} & \to & 0 & \to & R^{n-1} f_*\mathcal{L} & \to & 0. \end{array} \tag{5.2}$$

This implies that $R^{n-1} f_*\mathcal{L} = 0$, and for $1 \le i \le n-2$, the sheaf

$$R^i f_*\mathcal{L} \cong R^{i+1} f_*\mathcal{K} \tag{5.3}$$

has support of codimension $\ge i+2$ on X, since the fibre of f over the generic point of any $n-i-1$ dimensional subvariety has dimension $\le i$.

We also have $f_*\mathcal{L} \cong \mathcal{O}_X(D)$. Denote the reflexive hull of $\mathcal{L}$ by $\mathcal{O}_Y(D_Y)$. As $\mathcal{O}_X(D)$ is saturated, we have $f_*\mathcal{O}_Y(D_Y) \cong \mathcal{O}_X(D)$. Thus the exact sequence

$$0 \to \mathcal{L} \to \mathcal{O}_Y(D_Y) \to \mathcal{Q} \to 0 \tag{5.4}$$

gives rise to an injection

$$f_*\mathcal{Q} \hookrightarrow R^1 f_*\mathcal{L} \cong R^2 f_*\mathcal{K}. \tag{5.5}$$

Therefore $f_*\mathcal{Q}$ has support of codimension ≥ 3. Since $\mathcal{Q}$ is supported in codimension ≥ 2 on Y, its $R^i f_*$ has support of codimension $\ge i+3$.

By the above considerations, we can rewrite the Leray spectral sequence

$$\chi(Y, \mathcal{L}) = \sum\nolimits_i (-1)^i \chi(X, R^i f_*\mathcal{L}), \tag{5.6}$$

as

$$\chi(X, \mathcal{O}_X(D)) = \chi(Y, \mathcal{O}_Y(D_Y)) - \sum\nolimits_i (-1)^i \chi(X, R^i f_*\mathcal{L}) \tag{5.7}$$

$$- \sum\nolimits_i (-1)^i \chi(X, R^i f_*\mathcal{Q}) \tag{5.8}$$

$$= \chi(Y, \mathcal{O}_Y(D_Y)) + \mathcal{P}, \tag{5.9}$$

where $\mathcal{P}$ is a contribution from sheaves supported on the singular locus X of codimension ≥ 3. Our arguments imply that this contribution is local analytic in a stronger sense: it depends only on the analytic type of the orbifold locus of X and the divisor D. In fact the resolution of singularities can be chosen once and for all, and the constructions of $\mathcal{L}$ and $\mathcal{Q}$ are then universal, so that a local analytic isomorphism of quotient singularities gives a local analytic isomorphism between these sheaves.

The next step is to write D_Y and K_Y in terms of D and K_X and the exceptional divisors of the resolution. For a singular locus ${}^kS \in X$ of dimension k let $\{{}^{{}^kS}E_j\}$ be the exceptional hypersurfaces mapping surjectively to ${}^kS$ under f. Recall that every singular point ${}^0S$ (not necessarily isolated) is locally analytically isomorphic to $\mathbb{A}^n/\mu_r$, thus the configuration of $\{{}^{{}^0S}E_j\}$ depends only on the analytic singularity type of ${}^0S$. Then

$$f^*K_X = K_Y + N \text{ and } f^*D = D_Y + M, \tag{5.10}$$

where

$$N = \sum_{{}^kS} \sum_j^{{}^kS} \gamma_j^{{}^kS} E_j \text{ and } M = \sum_{{}^kS} \sum_j^{{}^kS} \beta_j^{{}^kS} E_j, \text{ with } {}^{{}^kS}\gamma_j, {}^{{}^kS}\beta_j \in \mathbb{Q}. \tag{5.11}$$

Here f^*D is by definition $\frac{1}{m}f^*(mD)$ for an integer m which makes mD Cartier.

Using Riemann–Roch [12, Appendix A] for the smooth n-fold Y we obtain

$$\chi(Y, \mathcal{O}_Y(mD_Y)) = \deg\left(\mathrm{ch}(\mathcal{O}_Y(mD_Y)) \cdot \mathrm{Td}(T_Y)\right)[n], \tag{5.12}$$

for

$$\begin{aligned} \mathrm{ch}(\mathcal{O}_Y(D_Y)) &= 1 + D_Y + \tfrac{1}{2}D_Y^2 + \tfrac{1}{6}D_Y^3 + \tfrac{1}{24}D_Y^4 + \cdots, \\ \mathrm{Td}(T_Y) &= 1 - \tfrac{1}{2}K_Y + \tfrac{1}{12}(K_Y^2 + c_2) - \tfrac{1}{24}K_Y c_2 - \\ &\quad \tfrac{1}{720}(K_Y^4 - 4K_Y^2 c_2 - 3c_2^2 + K_Y c_3 + c_4) + \cdots, \end{aligned} \tag{5.13}$$

where we set $c_i = c_i(T_Y)$, and $c_i = 0$ if $i > n$ and $()[n]$ is the component of degree n in the Chow ring $A(Y) \otimes \mathbb{Q}$. In particular, for $D_Y = 0$, we find that the degree n term in $\mathrm{Td}(T_Y)$ equals $\chi(\mathcal{O}_Y)$.

The projection formula gives

$$(f^*D)^i (f^*K_X)^{n-i} = D^i K_X^{n-i} \quad \text{for} \quad 0 \le i \le n. \tag{5.14}$$

Finally, using (5.7), Riemann–Roch for Y, (5.10), Definition (**??**) and the birational invariance of $\chi(\mathcal{O}_X) = \chi(\mathcal{O}_Y)$, we obtain

$$\chi(X, \mathcal{O}_X(D)) = \deg\Big(\mathrm{ch}(\mathcal{O}_X(D)) \cdot \mathrm{Td}(T_X)\Big)[n] + \mathcal{P} \tag{5.15}$$

$$+ \text{ intersection numbers involving } M \text{ or } N. \tag{5.16}$$

The intersection numbers involving M or N are of the form

$$M^i N^j (f^*D)^{j'} (f^*K_X)^{i'} c_{n-i-j-i'-j'}(Y) \quad \text{for} \quad i + j > 0. \tag{5.17}$$

Therefore they depend on the singularities of X and contribute to the orbifold basket part of the Riemann–Roch formula.

We conclude this section by noting that if (X, D) has a ${}_l\big(\frac{1}{r}(a_1, \dots, a_n)\big)$ singularity, then (X, mD) has a ${}_{\overline{ml}}\big(\frac{1}{r}(a_1, \dots, a_n)\big)$ singularity for all positive integers m. Here $\overline{}$ denotes the smallest residue mod r.

5.2. **The contribution from an orbifold point.** The analytically invariant contributions to RR from an orbifold locus, not necessarily isolated, are computed on a model that contains such orbifold locus. We begin by showing the existence of such projective varieties.

Proposition 5.1. *Given* $\frac{1}{r}(a_1,\dots,a_n)$*, we write* $a_{j_1},\dots,a_{j_{n-k}}$ *for the complement of the subset* $a_{i_1},\dots,a_{i_k}$*. Assume that all* $a_{i_1},\dots,a_{i_{n-1}},r$ *have no common factor. There exists a smooth projective n-fold* Z *together with an action of* μ_r *with the following properties: the action fixes a number of points on which a generator* $\varepsilon\in\mu_r$ *acts by*

$$\varepsilon\colon z_1,\dots,z_n \ \mapsto\ \varepsilon^{a_1}z_1,\dots,\varepsilon^{a_n}z_n. \tag{5.18}$$

If $\operatorname{hcf}(a_{i_1},\dots,a_{i_k},r)=\alpha_{i_1\dots i_k}\neq 1$*, these points lie on k-dimensional subvarieties which are fixed by* $\varepsilon^{\overline{\frac{r}{\alpha_{i_1\cdots i_k}}}}$*. Then* $\varepsilon^{\overline{\frac{r}{\alpha_{i_1\cdots i_k}}}}\in\mu_{\alpha_{i_1\dots i_k}}$ *acts in the normal direction of each k-fold by*

$$\varepsilon^{\overline{\frac{r}{\alpha_{i_1\cdots i_k}}}}: z_{j_1},\dots,z_{j_{n-k}} \ \mapsto\ \varepsilon^{\overline{\frac{r}{\alpha_{i_1\cdots i_k}}a_{j_1}}}z_{j_1},\dots,\varepsilon^{\overline{\frac{r}{\alpha_{i_1\cdots i_k}}a_{j_{n-k}}}}z_{j_{n-k}} \tag{5.19}$$

and freely away from the k-fold.

Proof. We imitate the proof of (8.4) in [YPG]. Choose an integer $l\ge n$ and consider action of μ_r on $\mathbb{P}^{l+n}(1,1,\dots,1)$ given by

$$x_1,x_2,x_3,x_4,\dots,x_{l+n+1} \ \mapsto\ \varepsilon^{a_1}x_1,\dots,\varepsilon^{a_n}x_n,x_{n+1},\dots,x_{l+n+1}. \tag{5.20}$$

This action fixes $\mathbb{P}^l=\{x_1=\cdots=x_n=0\}$ and acts in the normal direction by

$$x_1,\dots,x_n \ \mapsto\ \varepsilon^{a_1}x_1,\dots,\varepsilon^{a_n}x_n. \tag{5.21}$$

If $\operatorname{hcf}(a_{i_1},\dots,a_{i_k},r)=\alpha_{i_1\dots i_k}\neq 1$, the action is not free on $\mathbb{P}^{l+k}=\{x_{j_1}=\cdots=x_{j_{n-k}}=0\}$. This is fixed by $\varepsilon^{\overline{\frac{r}{\alpha_{i_1\cdots i_k}}}}$ with normal action

$$x_{j_1},\dots,x_{j_{n-k}}\mapsto \varepsilon^{\overline{\frac{r}{\alpha_{i_1\cdots i_k}}a_{j_1}}}x_{j_1},\dots,\varepsilon^{\overline{\frac{r}{\alpha_{i_1\cdots i_k}}a_{j_{n-k}}}}x_{j_{n-k}}. \tag{5.22}$$

Another locus on which the action might not be free, is $\{x_{n+1}=\cdots=x_{l+n+1}=0\}$. We avoid this locus by defining

$$X\subset\mathbb{P}^{l+n}/\mu_r, \tag{5.23}$$

as a complete intersection of l general very ample divisors. Let Z be the inverse image of X under the quotient $\mathbb{P}^{l+n}\to\mathbb{P}^{l+n}/\mu_r$. Such Z clearly satisfies the conditions in the proposition. $\square$

Let X be a projective threefold with a singularity of type $\frac{1}{r}(a_1,\dots,a_n)$ as described in Proposition 5.1.

Write $\pi\colon Z\to X$ for the quotient map and let $\mathcal{L}_l$ be the lth eigensheaf of the action of $\varepsilon\in\mu_r$ on $\pi_*\mathcal{O}_Z$. Then

$$\pi_*\mathcal{O}_Z=\bigoplus_{n=0}^{r-1}\mathcal{L}_l \tag{5.24}$$

implies

$$H^p(Z, \mathcal{O}_Z) = \bigoplus_{l=0}^{r-1} H^p(X, \mathcal{L}_l). \tag{5.25}$$

The group action on any $f \in \mathcal{L}_l$ is given by $\varepsilon(f) = \varepsilon^l \cdot f$, thus

$$\mathrm{Tr}(\varepsilon : H^p(Z, \mathcal{O}_Z)) = \sum_{l=0}^{r-1} h^p(X, \mathcal{L}_l) \cdot \varepsilon^l \tag{5.26}$$

and

$$\sum_p (-1)^p \mathrm{Tr}(\varepsilon : H^p(Z, \mathcal{O}_Z)) = \sum_{l=0}^{r-1} \chi(X, \mathcal{L}_l) \cdot \varepsilon^l. \tag{5.27}$$

In order to simplify the notation, denote $\sum_p (-1)^p \mathrm{Tr}(\varepsilon : H^p(Z, \mathcal{O}_Z))$ by A_ε. Then

$$\sum_{l=0}^{r-1} \chi(X, \mathcal{L}_l) \cdot \varepsilon^n = A_\varepsilon \quad \text{and} \quad \sum_{l=0}^{r-1} \chi(X, \mathcal{L}_l) = \chi(\mathcal{O}_Z). \tag{5.28}$$

The last two formulas can be considered as a linear system of r equations in $\chi(X, \mathcal{L}_l)$ and variable ε,

(5.29)

$$\begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \varepsilon & \varepsilon^2 & \dots & \varepsilon^{r-1} \\ 1 & \varepsilon^2 & \varepsilon^4 & \dots & \varepsilon^{2(r-1)} \\ \dots & \dots & \dots & \dots & \\ 1 & \varepsilon^{r-1} & \dots & \dots & \varepsilon \end{pmatrix} \begin{pmatrix} \chi(X, \mathcal{L}_0) \\ \chi(X, \mathcal{L}_1) \\ \chi(X, \mathcal{L}_2) \\ \vdots \\ \chi(X, \mathcal{L}_{r-1}) \end{pmatrix} = \begin{pmatrix} \chi(\mathcal{O}_Z) \\ A_\varepsilon \\ A_{\varepsilon^2} \\ \vdots \\ A_{\varepsilon^{r-1}} \end{pmatrix}.$$

Write $\chi(X, \mathcal{L}_0) = \chi(\mathcal{O}_X)$ and eliminate $\chi(\mathcal{O}_Z)$ from the solution. We end up with

$$\begin{aligned} \chi(X, \mathcal{L}_l) &= \chi(\mathcal{O}_X) + \frac{1}{r} \sum_{j=1}^{r-1} (\varepsilon^{-jl} - 1) A_{\varepsilon^j} \\ &= \chi(\mathcal{O}_X) + \frac{1}{r} \sum_{\varepsilon \in \mu_r} (\varepsilon^{-l} - 1) A_\varepsilon. \end{aligned} \tag{5.30}$$

We can compute A_ε for all $\varepsilon \in \mu_r$ by the following proposition.

Proposition 5.2. *Let ${}^kZ$ be a k-dimensional subvariety in Z fixed by $\varepsilon \in \mu_s$, which acts in the normal direction of ${}^kZ$ by*

$$z_1, \dots, z_{n-k} \;\mapsto\; \varepsilon^{b_1} z_1, \dots, \varepsilon^{b_{n-k}} z_{n-k} \tag{5.31}$$

and freely away from ${}^kZ$. Let m_j of the b_js be the same. Then A_ε is a sum over degree k intersections involving the Chern classes of the tangent and normal bundles $T_{{}^kZ}$, $\mathcal{N}_{{}^kZ}$. The coefficients of this sum are rational functions in ε^{b_j} with symmetric numerators and denominators $\prod_j (1-\varepsilon^{b_j})^{m_j+l_j}$, where $0 \le l_j \le k$ and $\sum_j l_j \le k$.

Proof. The normal bundle $\mathcal{N}_{{}^kZ} = \oplus_j \mathcal{N}_j$ decomposes according to the eigenvalues ε^{b_i}. In our notation $\mathcal{N}_j$ has rank m_j. Note that $\sum_j m_j = n-k$. Each $\mathcal{N}_j$ might decompose further. Denote x_{j_t} the Chern classes of $\mathcal{N}_j$. In particular, if $\mathcal{N}_j$ decomposes into m_j line bundles, then x_{j_t} is the 1st Chern class of the tth line bundle; on the other hand, if $\mathcal{N}_j$ is indecomposable, then x_{j_t} is its tth Chern class. By the Atiyah–Singer–Segal equivariant RR formula [3], [4, p. 565] we get

$$A_\varepsilon = \deg \left\{ \frac{1 \cdot \mathrm{Td}(T_{{}^kZ})}{\prod_j \prod_{t=1}^{m_j} (1 - \varepsilon^{-b_j} e^{-x_{j_t}})} \right\}_k . \tag{5.32}$$

It is an exercise to compute the power series expansion below for $a \neq 1$

$$\begin{aligned} \frac{1}{1-ae^{-x}} &= \frac{1}{1-a} - \frac{a}{(1-a)^2}x + \frac{a(1+a)}{2(1-a)^3}x^2 \\ &\quad - \frac{a(1+4a+a^2)}{6(1-a)^4}x^3 + \frac{a(1+11a+11a^2+a^3)}{24(1-a)^5}x^4 + \cdots . \end{aligned} \tag{5.33}$$

The proposition follows by writing down the degree k elements in the Chow ring $A({}^kZ)$ of the product

$$\begin{aligned} \mathrm{Td}(T_{{}^kZ}) \;=\; & 1 + \tfrac{1}{2}c_1 + \tfrac{1}{12}(c_1^2 + c_2) + \tfrac{1}{24}c_1c_2 - \\ & \tfrac{1}{720}(c_1^4 - 4c_1^2c_2 - 3c_2^2 - c_1c_3 + c_4) + \cdots , \end{aligned} \tag{5.34}$$

where $c_i = c_i(T_{{}^kZ})$, for $1 \le i \le k$, and the power series expansions of all $\frac{1}{1-\varepsilon^{-b_j}e^{-x_{j_t}}}$. Observe that the rational coefficients of the latter expansions have symmetric numerators. $\square$

When $\varepsilon^{a_i} \neq 1$ for all $i = 1, \ldots, n$, the fixed locus of ε is a number of points. In this case it is a direct corollary of Proposition 5.2, that for each point

$$A_\varepsilon = \frac{1}{(1-\varepsilon^{-a_1}) \cdots (1-\varepsilon^{-a_n})}. \tag{5.35}$$

On the other hand, if $\alpha_{i_1 \ldots i_k}$ corresponding to the subset $\{i_1, \ldots, i_k\} \subset \{1, \ldots, n\}$ is different from 1, then $\varepsilon \in \mu_{\alpha_{i_1 \ldots i_k}}$ fixes a k-fold whose generic points are orbifold points of type $\frac{1}{\alpha_{i_1 \ldots i_k}}(a_{j_1}, \ldots, a_{j_{n-k}})$.

We can split the sum over $\{\varepsilon \in \mu_r\}$ into subsums over disjoint sets

$$\begin{aligned} &\{\varepsilon \in \mu_r \text{ s.t. } \varepsilon^{a_i} \neq 1 \text{ for all } i = 1, \ldots, n\} \quad \text{and} \\ &\{\varepsilon \in \mu_r \text{ s.t. } \varepsilon^{a_{i_1}} = \cdots = \varepsilon^{a_{i_k}} = 1\} = \{\varepsilon \in \mu_{\alpha_{i_1 \ldots i_k}}\}. \end{aligned} \tag{5.36}$$

This rewrites (5.30) as

$$\begin{aligned} \chi(X, \mathcal{L}_l) &= \chi(\mathcal{O}_X) \\ &+ \frac{1}{r} \sum_{\substack{\varepsilon \in \mu_r \\ \varepsilon^{a_i} \neq 1 \forall i = 1, \ldots, n}} \frac{\varepsilon^{-l} - 1}{(1-\varepsilon^{-a_1}) \cdots (1-\varepsilon^{-a_n})} + \frac{1}{r} \sum_{\varepsilon \in \mu_{\alpha_{i_1 \ldots i_k}}} (\varepsilon^{-l} - 1) A_\varepsilon, \end{aligned} \tag{5.37}$$

where the A_ε were described in Proposition 5.2.

We view $\chi(\mathcal{O}_X)$ as the RR-type expression for the pair (X, L_n), where $L_n = 0 \in$ Div$X \otimes \mathbb{Q}$ is the $\mathbb{Q}-$divisor corresponding to $\mathcal{L}_n$. The rest of the formula is a sum of contributions coming from

- points of type $\frac{1}{r}(a_1, \dots, a_n)$, and
- k-folds of generic type $\frac{1}{\alpha_{i_1 \dots i_k}}(a_{j_1}, \dots, a_{j_{n-k}})$,

which are the singularities of (X, L_n).

Remark 5.3. The attentive reader will notice that the last two sections use different approaches to obtain the same result. First we considered resolution of singularities $Y \to X$. To compute the contributions to RR from the singularities explicitly, we would need to relate in a compact way all the intersection numbers involving exceptional divisors. Second we constructed an auxiliary μ_r cover of a model variety and used equivariant RR. This only gave us the shape of the contributions to RR arising from the singularities. For a more precise description on a polarized orbifold X with k dimensional singular locus ${}^kX$, we would need to consider a $\mu_r \underbrace{\oplus \cdots \oplus}_{k+1} \mu_r$ cover $\pi\colon Z \to X$ and rewrite our proofs for the eigensheaves of this action on $\pi_* \mathcal{O}_Z$. This way the contributions to RR depend, not only on the singularity type of ${}^kX$, but also on the embedding of ${}^kX$ in X. The latter information determines the splitting of the normal bundle of ${}^kZ$ over ${}^kX$.

5.3. **Application to isolated orbifold points.** Note that a point $P \in X$ of type $\frac{1}{r}(a_1, \dots, a_n)$ is isolated if and only if $\operatorname{hcf}(a_i, r) = 1$ for all $i = 1, \dots, n$. This means that for all i and $1 \neq \varepsilon \in \mu_r$ holds $\varepsilon^{a_i} \neq 1$. The following theorem is a direct corollary of 5.1 and 5.2.

Theorem 5.4. *Let (X, D) be a pair consisting of a normal projective n-fold and a $\mathbb{Q}$-Cartier divisor, which is quasismooth and well formed. Assume further that the singularities of (X, D) consist of the isolated points $Q \in X$ of type ${}_l\big(\frac{1}{r}(a_1, \dots, a_n)\big)$. Then for all positive integers m,*

$$\chi(X, \mathcal{O}_X(mD)) = \deg\left(\operatorname{ch}(\mathcal{O}_X(mD)) \cdot \operatorname{Td}(T_X)\right)[n] + \sum_Q c_Q(mD), \tag{5.38}$$

where

$$c_Q(mD) = \frac{1}{r} \sum_{1 \neq \varepsilon \in \mu_r} \frac{\varepsilon^{-lm} - 1}{(1 - \varepsilon^{-a_1}) \cdots (1 - \varepsilon^{-a_n})}. \tag{5.39}$$

6. Proof of the main theorem

In this section we restrict to (X, D) with a basket of isolated orbifold points $\mathcal{B} = \{Q \in X \text{ of type } \frac{1}{r}(b_1, \dots, b_n)\}$. The main point of the decomposition formula is that the contribution from the isolated orbifold points depends only on the type $\frac{1}{r}(b_1, \dots, b_n)$ of the polarized orbifold point and the canonical weight (compare the K3 surfaces and Fano 3-folds in Section 3).

// Omit – this is well known
We start with a technical lemma.

Lemma 6.1. *Let $R(m)$ be a polynomial of degree n in variable m. Also let $\{q_i\}_{i\in\mathbb{Z}}$ be a sequence that repeats periodically with period r.*

(a) *Assume that there exists an integer $m_0 \geq 0$, such that the power series $\sum_{m\geq 0}\rho_m t^m$ satisfies $\rho_m = R(m)$ for all $m \geq m_0$. Then $(1-t)^{n+1}\left(\sum_{m\geq 0}\rho_m t^m\right)$ is a polynomial in t of degree at most m_0+n+1.*
(b) $(1-t)^{n+1}\left(\sum_{m\in\mathbb{Z}} R(m)t^m\right) = 0$.
(c) $(1-t^r)\left(\sum_{i\in\mathbb{Z}} q_i t^i\right) = 0$.

Proof. (a) Multiplying our power series by $1-t$ gives $(1-t)\left(\sum_{m\geq 0}\rho_m t^m\right) = \cdots + (R(m_0+1)-R(m_0))t^{m_0+1} + (R(m_0+2)-R(m_0+1))t^{m_0+2}+\cdots$. For all $m \geq m_0+1$ in this power series, the mth coefficient is a polynomial in m of degree $n-1$, namely $R(m)-R(m-1)$. Repeating this step $n+1$ times, by induction implies (a).

(b) The coefficients of $(1-t)\left(\sum_{m\in\mathbb{Z}} R(m)t^m\right) = \sum_{m\in\mathbb{Z}}(R(m)-R(m-1))t^m$ are polynomials of degree $n-1$. Repeat this $n+1$ times to finish the proof of (b).

(c) This is obvious because

$$\sum_{i\in\mathbb{Z}} q_i t^i = \left(q_0 + q_1 t + \cdots + q_{r-1}t^{r-1}\right)\left(\sum_{i\in\mathbb{Z}} t^{ri}\right). \quad \square$$

We divide the proof of Theorem 1.1 into the following claims and propositions, analysing the shape and symmetry of the components in the decomposition of $P(t)$.

Under our assumptions we equate $\mathcal{O}_X(mD) = \mathcal{O}_X(m)$ with isolated orbifold points of type $\frac{1}{r}(b_1,\dots,b_n)$. By Theorem 5.4, we have

$$\chi(X,\mathcal{O}_X(m)) = \deg\left(\operatorname{ch}(\mathcal{O}_X(m))\cdot \operatorname{Td}(T_X)\right)[n] + \sum_Q c_Q(m), \tag{6.1}$$

where

$$c_Q(m) = \frac{1}{r}\sum_{1\neq\varepsilon\in\mu_r}\frac{\varepsilon^{-m}-1}{(1-\varepsilon^{b_1})\cdots(1-\varepsilon^{b_n})}. \tag{6.2}$$

Here we replace ε by ε^{-1} for clarity. Recall that

$$\chi(X,\mathcal{O}_X(m)) = h^0(X,\mathcal{O}_X(m)) + (-1)^n h^n(X,\mathcal{O}_X(m)), \tag{6.3}$$

since X is projectively Gorenstein. Serre duality gives $h^n(X, \mathcal{O}_X(m)) = h^0(X, \mathcal{O}_X(k_X - m))$, which $= 0$ for $m > k_X$. Therefore

$$\begin{aligned}P(t) &= \sum_{m\geq 0} h^0(X, \mathcal{O}_X(m))t^m \\ &= \sum_{m=0}^{k_X} \left(\deg\left(\mathrm{ch}(\mathcal{O}_X(m)) \cdot \mathrm{Td}(T_X)\right)[n] + (-1)^{n+1} h^0(X, \mathcal{O}_X(k_X - m))\right) t^m \\ &\quad + \sum_{m>k_X} \deg\left(\mathrm{ch}(\mathcal{O}_X(m)) \cdot \mathrm{Td}(T_X)\right)[n] t^m + \sum_Q \sum_{m\geq 0} c_Q(m)t^m.\end{aligned}$$

Claim 6.2. *For an orbifold point Q of type $\frac{1}{r}(b_1, \cdots, b_n)$ we have*

$$\sum_{m\geq 0} c_Q(m)t^m = \frac{\mathrm{InvMod}\left(\prod_{i=1}^n (1 - t^{a_i}), \frac{1-t^r}{1-t}, 0\right)}{1 - t^r}. \tag{6.4}$$

Proof. Since the sum is periodic in r, we have

$$(1 - t^r) \left(\sum_{m\geq 0} c_Q(m)t^m\right) = \frac{1}{r} \sum_{i=0}^{r-1} \left(\sum_{1\neq\varepsilon\in\mu_r} \frac{\varepsilon^{-i} - 1}{(1 - \varepsilon^{a_1}) \cdots (1 - \varepsilon^{a_n})}\right) t^i. \tag{6.5}$$

In Dedekind sum notation, this splits into

$$\sum_{i=0}^{r-1} \sigma_{r-i} t^i - \frac{1}{r} \left(\sum_{1\neq\varepsilon\in\mu_r} \frac{1}{(1 - \varepsilon^{a_1}) \cdots (1 - \varepsilon^{a_n})}\right) \frac{1 - t^r}{1 - t}. \tag{6.6}$$

Theorem 2.6 completes the proof. □

Claim 6.3. *$F(t) := P(t) - \sum_Q \sum_{m\geq 0} c_Q(m)t^m$ is of the form $\frac{f(t)}{(1-t)^{n+1}}$, where $f(t)$ is a polynomial of degree $k_X + n + 1$.*

Proof. Indeed, for $m > k_X$, the coefficients of $F(t)$ are equal to

$$\deg\left(\mathrm{ch}(\mathcal{O}_X(m)) \cdot \mathrm{Td}(T_X)\right)[n],$$

which are polynomials of degree n in variable m. Lemma 6.1 (a) proves the claim. □

Before we move on, we introduce a global property of the Hilbert series, which comes from the projectively Gorenstein property of the variety. We call $P(t)$ *Gorenstein symmetric* if it satisfies the following proposition.

Proposition 6.4. $P(t) = (-1)^{n+1} t^{k_X} P\left(\frac{1}{t}\right)$.

Proof. Recall that $\chi(X, \mathcal{O}_X(m)) = h^0(X, \mathcal{O}_X(m)) + (-1)^n h^0(X, \mathcal{O}_X(k_X - m))$ and $h^0(X, \mathcal{O}_X(i)) = 0$ for $i < 0$. Summing the coefficients of $P(t)$ and $(-1)^n t^{k_X} P\left(\frac{1}{t}\right)$ corresponding to the same power of t yields

$$\sum_{m\in\mathbb{Z}} \chi(X, \mathcal{O}_X(m))t^m = P(t) + (-1)^n t^{k_X} P\left(\frac{1}{t}\right). \tag{6.7}$$

On the other hand, by the Riemann–Roch formula (6.1),

$$\sum_{m\in\mathbb{Z}} \chi(X, \mathcal{O}_X(m))t^m = \sum_{m\in\mathbb{Z}} \deg\left(\mathrm{ch}(\mathcal{O}_X(m)) \cdot \mathrm{Td}(T_X)\right)[n]t^m + \sum_{m\in\mathbb{Z}}\sum_{Q} c_Q(m)t^m. \tag{6.8}$$

Multiply this by $(1-t)^{n+1}(1-t^r)$ and use Lemma 6.1 (b) and (c) to prove that it equals to 0. □

Next we are going to describe P_{orb}. In Claim 6.2 we have already got an expression corresponding to each singular point Q, namely $\frac{g_Q(t)}{1-t^r}$. Now we shift the support of $(1-t)^n g_Q(t)$ into $\left[\left\lfloor \frac{c}{2}\right\rfloor + 1,\ \left\lfloor \frac{c}{2}\right\rfloor + r - 1\right]$. This way we obtain the numerator $B(t)$ in Theorem 1.1.

Claim 6.5. *For each singular point Q of type $\frac{1}{r}(a_1, \cdots, a_n)$ there exists a unique $B_Q(t)$ supported in $[\left\lfloor \frac{c}{2}\right\rfloor + 1, \left\lfloor \frac{c}{2}\right\rfloor + r - 1]$, such that $\frac{B_Q(t)}{(1-t)^n(1-t^r)} = \frac{g_Q(t)}{(1-t^r)} + \frac{C_Q(t)}{(1-t)^{n+1}}$ for some polynomial $C(t)$. Moreover, $P_{\mathrm{orb}}(t) = \frac{B_Q(t)}{(1-t)^n(1-t^r)}$ is also Gorenstein symmetric:*

$$P_{\mathrm{orb}}(t) = (-1)^{n+1} t^{k_X} P_{\mathrm{orb}}\left(\frac{1}{t}\right). \tag{6.9}$$

Proof. Since $B(t)$ and $(1-t)^n g(t)$ are both inverses of $\prod_{i=1}^n \frac{1-t^{a_i}}{1-t}$ modulo $\frac{1-t^r}{1-t}$, there exists a polynomial $C(t)$ satisfying

$$B_Q(t) = g_Q(t)(1-t)^n + C_Q(t)(1 + t + t^2 + \cdots + t^{r-1}). \tag{6.10}$$

So $g_Q(t)(1-t)^n$ can be rewritten in a unique way supported in $[\left\lfloor \frac{c}{2}\right\rfloor + 1, \left\lfloor \frac{c}{2}\right\rfloor + r - 1]$. Indeed, $\mathbb{Q}[t]/(1 + t + \cdots + t^{r-1})$ is $r-1$ dimensional over $\mathbb{Q}$ and t is invertible, therefore also $\mathbb{Q}[t, t^{-1}]/(1 + t + \cdots + t^{r-1})$ is $r-1$ dimensional.

By Corollary 2.8 the orbifold contribution equals

$$P_{\mathrm{orb}}(t) = \frac{t^{\left\lfloor \frac{c}{2}\right\rfloor + 1} \sum_{j=0}^{r-2} \Theta_j t^j}{(1-t)^n(1-t^r)}. \tag{6.11}$$

The Gorenstein symmetry follows immediately from Exercise 2.10. □

Claim 6.5 shows that P_{orb} consists of the contribution from the singular point and some correction, which makes it Gorenstein symmetric. Now denote the remaining $P(t) - \sum_Q P_{\mathrm{orb}}(t)$ by $P_I(t)$ and call it the *initial term*. The following proposition tells us how to determine the initial term explicitly.

Proposition 6.6. *Let $P(t) = \sum_{m\geq 0} P_m t^m$ be the Hilbert series of X. The initial term $P_I(t)$ is uniquely determined by the first $\lfloor \frac{c}{2} \rfloor$ coefficients of $P(t)$ and is of the form $\frac{A(t)}{(1-t)^{n+1}}$. Its numerator is $A(t) = \sum_{i=0}^{c} A_i t^i$, with*

$$A_i = A_{c-i} = \sum_{j=0}^{i} (-1)^{i-j} \binom{n+1}{i-j} P_j \quad \textit{for } i \in [0, \lfloor \tfrac{c}{2} \rfloor]. \tag{6.12}$$

Proof. From the previous constructions, one sees that $P_I(t)$ is of the form $\frac{A(t)}{(1-t)^{n+1}}$. Note that $P_I(t)$ is also Gorenstein symmetric, of weight k_X, since both $P(t)$ and $P_{\mathrm{orb}}(t)$ are.

If $c \geq 0$, all the series $P_{\mathrm{orb}}(t)$ start from degree $\lfloor \frac{c}{2} \rfloor + 1$, so that $P_I(t)$ and $P(t)$ are equal up to and including degree $\lfloor \frac{c}{2} \rfloor$. Write $P(t)(1-t)^{n+1} = \sum_{i\geq 0} b_i t^i$ and compare coefficients. Considering the symmetry, we have $A_i = b_i = A_{c-i}$ for $i \in [0, \lfloor \frac{c}{2} \rfloor]$. Of course we take $\binom{n+1}{k} = 0$ for $k > n+1$.

If $c < 0$, the same analysis shows that $P_I(t) = 0$. □

7. Computer pseudocode

We include the computer algorithm in `Wolfram Mathematica` that show how to compute ice cream functions.

Fix $a_1, \dots, a_n, r \in \mathcal{N}$ and $\gamma \in \mathbb{Z}$. We will calculate:

(I) the inverse of $\prod_{i=1}^{n}(1 - t^{a_i})$ modulo $\frac{1-t^r}{1-t} = 1 + t + \cdots + t^{r-1}$ as a polynomial in the range

$$\left[\gamma, \gamma+1, \dots, \gamma + r - 2 - \deg \operatorname{hcf}\left(\prod_{i=1}^{n}(1-t^{a_i}), \frac{1-t^r}{1-t}\right)\right].$$

(II) the inverse of $\prod_{i=1}^{n} \frac{1-t^{a_i}}{1-t}$ modulo $\frac{1-t^r}{1-t} = 1 + t + \cdots + t^{r-1}$ as a polynomial in the range

$$\left[\gamma, \gamma+1, \dots, \gamma + r - 2 - \deg \operatorname{hcf}\left(\prod_{i=1}^{n}(1-t^{a_i}), \frac{1-t^r}{1-t}\right)\right].$$

In the case of Calabi–Yau 3-folds we also need "N_C-Num" and "D_C-Num", where

(III) N_C-Num $\times \frac{(1-t^a)(1-t^{r-a})^2}{(1-t)^3} \equiv 1 + t^{r-a}$ modulo $\frac{1-t^r}{1-t}$ and N_C-Num is a polynomial with support in the range $[3, \dots, r]$.

(IV) D_C-Num is a polynomial with support in the range $[3, \dots, r]$, evaluated in Lemma 4.3 .

The command **PolynomialGCD**[] runs the Euclidean algorithm on polynomials. We first calculate the highest common factor **hcf**:

$$\mathbf{hcf}[\mathbf{a1},\dots,\mathbf{an},\mathbf{r}] := \mathbf{PolynomialGCD}\big[\mathbf{PolynomialQuotient}[(\mathbf{1}-\mathbf{t}^{\mathbf{a1}})\cdots(\mathbf{1}-\mathbf{t}^{\mathbf{an}}),(\mathbf{1}-\mathbf{t})^{\mathbf{n}},\mathbf{t}],\ \mathbf{PolynomialQuotient}[\mathbf{1}-\mathbf{t}^{\mathbf{r}},\mathbf{1}-\mathbf{t},\mathbf{t}]\big]. \tag{7.1}$$

The inverse (I) with the required support is $t^\gamma(\mathbf{I})\mathbf{inv}$, where

$$(\mathbf{I})\mathbf{inv}[\mathbf{a1},\dots,\mathbf{an},\mathbf{r},\gamma] := \mathbf{PolynomialExtendedGCD}\big[\mathbf{Polynomial}[\mathbf{t}^{\gamma}(\mathbf{1}-\mathbf{t}^{\mathbf{a1}})\cdots(\mathbf{1}-\mathbf{t}^{\mathbf{an}}),\mathbf{t}],\ \mathbf{PolynomialQuotient}[\mathbf{1}-\mathbf{t}^{\mathbf{r}},\mathbf{hcf},\mathbf{t}]\big].$$

The inverse (II) with the required support is $t^\gamma(\mathbf{II})\mathbf{inv}$, where

$$(\mathbf{II})\mathbf{inv}[\mathbf{a1},\dots,\mathbf{an},\mathbf{r},\gamma] := \mathbf{PolynomialExtendedGCD}\ \mathbf{PolynomialQuotient}[\mathbf{t}^{\gamma}(\mathbf{1}-\mathbf{t}^{\mathbf{a1}})\cdots(\mathbf{1}-\mathbf{t}^{\mathbf{an}}),(\mathbf{1}-\mathbf{t})^{\mathbf{n}},\mathbf{t}],\ \mathbf{PolynomialQuotient}[\mathbf{1}-\mathbf{t}^{\mathbf{r}},\mathbf{hcf},\mathbf{t}].$$

The N_C–**Num** in (III) equals:

$$\mathbf{t}^{\mathbf{3}}\left((\mathbf{II})\mathbf{inv}[\mathbf{a},\mathbf{r}-\mathbf{a},\mathbf{r}-\mathbf{a},\mathbf{r},\mathbf{3}]+(\mathbf{II})\mathbf{inv}[\mathbf{a},\mathbf{r}-\mathbf{a},\mathbf{r}-\mathbf{a},\mathbf{r},\mathbf{3}-\mathbf{r}+\mathbf{a}]\right).$$

The D_C–**Num** in (IV) equals:

Consider the part of $\mathbf{t}\left(\mathbf{A}-\mathbf{A}[[\mathbf{0}]]\frac{\mathbf{1}-\mathbf{t}^{\mathbf{r}}}{\mathbf{1}-\mathbf{t}}-\mathbf{A}[[\mathbf{1}]]\mathbf{t}\frac{\mathbf{1}-\mathbf{t}^{\mathbf{r}}}{\mathbf{1}-\mathbf{t}}\right)$ with support in $[3,\dots,\lceil\frac{r}{2}\rceil+1]$ and extend it to the unique symmetric polynomial supported in $[3,\dots,r]$. Here $\mathbf{A}[[\mathbf{0}]],\mathbf{A}[[\mathbf{1}]]$ are the coefficients at $1,t$ in $\mathbf{A}$, where

$$\mathbf{A} := (\mathbf{1}-\mathbf{t})^{\mathbf{3}}\mathbf{Derivative}[\mathbf{t}(\mathbf{I})\mathbf{inv}[\mathbf{a},\mathbf{r}-\mathbf{a},\mathbf{r},\mathbf{1}]]. \tag{7.2}$$

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Proof. [YPG], (8.5), (3–5) treats the σ_i as follows (with a spare factor of r): Y is a n-fold with an action of μ_r having N isolated fixed points, each of the same type $\frac{1}{r}(b_1, \dots, b_n)$. Write $\pi\colon Y \to X$ for the quotient morphism, and $\mathcal{L}_i$ for the ith eigensheaf on X (as in (1.10)). Then by the Lefschetz fixed point formula, the r quantities

$$\sigma_i = \frac{1}{N}\chi(\mathcal{L}_i) - \frac{1}{Nr}\chi(\mathcal{O}_Y) \tag{7.3}$$

are determined by the nondegenerate system of r linear equations

$$\sum_{i=0}^{r-1} \sigma_i \varepsilon^i = \frac{1}{\prod_j (1 - \varepsilon^{-b_j})} \quad \text{for all } \varepsilon \in \mu_r \setminus 1, \quad \text{and} \quad \sum \sigma_i = 0. \tag{7.4}$$

The second equation just says $\sum \sigma_i = \frac{1}{N}\big(\sum \chi(\mathcal{L}_i) - \chi(\mathcal{O}_Y)\big) = 0$.

We can apply this equation equally well to ε^{-1}; reordering the sum and taking account of $\varepsilon^r = 1$ gives

$$\frac{1}{\prod_j (1 - \varepsilon^{b_j})} = \sum_{i=0}^{r-1} \sigma_i \varepsilon^{r-i} = \sum_{i=1}^{r} \sigma_{r-i} \varepsilon^i \quad \text{for all } \varepsilon \in \mu_r \setminus 1, \tag{7.5}$$

Now consider the polynomial $B(t) = A(t) \sum_{i=1}^r \sigma_{r-i} t^i$. If we substitute $t = \varepsilon$ for $\varepsilon \in \mu_r \setminus 1$ in B and use (7.5) with the same value of ε, we get

$$B(\varepsilon) = A(\varepsilon) \sum_{i=1}^{r} \sigma_{r-i} \varepsilon^i = \frac{A(\varepsilon)}{\prod_j (1 - \varepsilon^{b_j})} = 1. \tag{7.6}$$

This holds for every root ε of F, so $B(t) - 1$ is divisible by F, that is,

$$A(t) \sum_{i=1}^{r} \sigma_{r-i} t^i \equiv 1 \mod F. \quad \square \tag{7.7}$$

To do. In 1.2, what does “precise shape” mean?