# Orbifold Riemann–Roch and plurigenera

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**Abstract**

I give a general formula for the Hilbert series of a polarised $n$-dimensional orbifold (for example, with isolated orbifold points). The result comes from orbifold RR, and so ultimately from equivariant RR (the Atiyah–Singer Lefschetz trace formula); however, the formula is organised so that no Chern or Todd classes appear explicitly, and no Dedekind sums. The formula reduces much of my work over 20 years to a few lines of computer algebra.

For dramatic effect, I state a simple case of the theorem first, leaving definitions and explanations for later.

**Theorem 1** *Let $X, \bigoplus \mathcal{O}_X(i)$ be a simply polarised $n$-fold with $n \geq 2$. Assume that $X$ is projectively Gorenstein with canonical weight $k_X$ and has a basket of isolated orbifold points $\mathcal{B} = \left\{\frac{1}{r}(a_1, \dots, a_n)\right\}$ as its only singularities.*

*Then the Hilbert series $P_X(t) = \sum_{n\geq 0} h^0(X, \mathcal{O}_X(n))t^n$ of $X$ is*

$$P_X(t) = P_I(t) + \sum_{\mathcal{B}} P_{\text{orb}}(\tfrac{1}{r}(a_1, \dots, a_n), k_X), \tag{1}$$

*with* initial term $P_I(t)$ *and* orbifold terms $P_{\text{orb}}(\frac{1}{r}(a_1, \dots, a_n), k_X)$ *characterised as follows:*

- *The initial term $P_I(t) = \frac{A(t)}{(1-t)^{n+1}}$ has denominator $(1-t)^{n+1}$, and numerator $A(t)$ a Gorenstein symmetric polynomial of degree the* coindex *$c = k_X + n + 1$ of $X$, so that $P_I(t)$ equals $P(t)$ up to degree $\left[\frac{c}{2}\right]$.*

- *Each orbifold term $P_{\text{orb}}(\frac{1}{r}(a_1, \dots, a_n), k_X) = \frac{B(t)}{(1-t)^n(1-t^r)}$ has denominator $(1-t)^n(1-t^r)$, and numerator $B(t)$ the unique Laurent polynomial supported in $\left[\left[\frac{c}{2}\right] + 1, \left[\frac{c}{2}\right] + r - 1\right]$ which is*

$$\textit{the inverse modulo} \quad \frac{1-t^r}{1-t} = 1 + t + \cdots + t^{r-1} \quad \textit{of} \quad \prod \frac{1-t^{a_i}}{1-t}.$$

The initial term $P_I(t)$ determines and is determined by the first $\left[\frac{c}{2}\right]$ plurigenera, and is 0 if $c < 0$. The coefficients of the orbifold terms are general Dedekind sums, but are determined by conceptually very simple *ice cream functions* (see Example 2; they are given by easy computer algebra routines.

**Example 2 ("Ice cream on Wednesdays, Fridays and Sundays")**
The step function $i \mapsto [3i/7]$ is familiar. As Hilbert series, it gives

$$P(t) := \sum_{i\geq 0}[3i/7]t^i = 0 + 0t + 0t^2 + t^3 + t^4 + 2t^5 + 2t^6 + 3t^7 + \cdots \qquad (2)$$

This series takes the closed form

$$P(t) = \frac{t^3 + t^5 + t^7}{(1-t)(1-t^7)}. \qquad (3)$$

In fact, since $[3i/7]$ increases cumulatively by 1 at $i = 3, 5, 7 \bmod 7$, it follows that $(1-t)P(t)$ is the sum of $t^i$ taken over the jumps

$$(1-t)P(t) = \sum_{i\geq 0} t^i = t^3 + t^5 + t^7 + t^{10} + \cdots \qquad (4)$$

repeating periodically with period 7. Multiplying by $(1-t^7)$ cuts this down to the first week's ration of ice cream.

The numerator $t^3+t^5+t^7$ is the inverse of $(1-t^5)/(1-t) = 1+t+t^2+t^3+t^4$ modulo $(1-t^7)/(1-t) = 1+t+t^2+t^3+t^4+t^5+t^6+t^7$ (here 5 is the inverse of 3 mod 7). Proof: the product $(1+t+t^2+t^3+t^4)(t^3+t^5+t^7)$ consists of 15 terms, distributed equitably among the 7 powers of $t$ modulo $t^7$, except for $3\times t^7$, so that $(1+t+t^2+t^3+t^4)(t^3+t^5+t^7) =$

$$\begin{array}{ccccccccccccccccccc}
1 & + & t & + & t^2 & + & t^3 & + & t^4 & & & & & & & & & & \\
 & & & & & + & t^3 & + & t^4 & + & t^5 & + & t^6 & + & t^7 & & & & \\
 & & & & & & & & & + & t^5 & + & t^6 & + & t^7 & + & t^8 & + & t^9
\end{array}$$

$$\begin{aligned}
&\equiv 3 + 2t + 2t^2 + \cdots + 2t^6 \bmod 1 - t^7 \\
&\equiv 1 \bmod 1 + t + t^2 + t^3 + t^4 + t^5 + t^6.
\end{aligned}$$

There are several other meaningful expressions for $P(t)$: working modulo $1+t+t^2+t^3+t^4+t^5+t^6$, one can view the bounty $t^3+t^5+t^7$ as famine

$-t - t^2 - t^4 - t^6$ ("no ice cream on Mondays, Tuesdays, Thursdays and Saturdays"), that is,

$$P(t) = \frac{t^3 + t^5 + t^7}{(1-t)(1-t^7)} = \frac{1}{(1-t)^2} + \frac{-t - t^2 - t^4 - t^6}{(1-t^7)}. \tag{5}$$

Either of these functions can be shifted up or down, e.g., to give

$$\frac{t^{-4} + t^{-2} + 1}{(1-t)(1-t^7)} \quad \text{or} \quad \frac{-t^{-1} - t - t^2 - t^4}{(1-t)(1-t^7)}, \quad \text{etc.,} \tag{6}$$

"ice cream rations start one week (or one day) before term".

Or the macroeconomic view is that $[3i/7]$ is the linear function $3i/7$ with periodic corrections, giving

$$P(t) = \frac{3}{7} \times \frac{1}{(1-t)^2} + \frac{-3/7t - 6/7t^2 - 2/7t^3 - 5/7t^4 - 1/7t^5 - 4/7t^6}{1 - t^7}. \tag{7}$$

The coefficients here are Dedekind sums. We will see that the general $P_{\text{orb}}$ and general Dedekind sums are obtained by minor variations on this simple calculation.

# 1 Introduction

## 1.1 Terminology

**Definition 3** A *simply polarised orbifold with isolated orbifold points* is a variety $X$ polarised by a sheaf of graded algebras $\bigoplus \mathcal{O}_X(i)$ satisfying:

- $X$ is a projective $n$-fold over a field $k$ (e.g., $k = \mathbb{C}$), and $\mathcal{O}_X(m)$ is an ample invertible sheaf for some $m > 0$;
- $X$ has at most isolated orbifold singularities $\frac{1}{r}(a_1, \dots, a_n)$, and locally at each point, each $\mathcal{O}(i)$ is isomorphic to the $i$th eigensheaf of the $\mu_r$ action.

I say *simply* polarised to mean $\mathbb{Z}$-graded or $\mathbb{N}$-graded (as opposed to lattice polarised or graded by a more complicated semigroup).

I assume in this introduction that orbifold behaviour occurs at isolated points in codimension $\geq 2$, so $n = \dim X \geq 2$. (The methods also apply to orbifold behaviour in codimension 1 or 0 after some elementary stacky preliminaries; see 1.3 for orbifold curves.) Then the $\mathcal{O}(i)$ are divisorial sheaves, and $\mathcal{O}(i) = \mathcal{O}_X(iA)$ for an ample Weil divisor $A$. In this case the *graded structure sheaf* $\bigoplus \mathcal{O}_X(i)$ is specified by $A$ or $\mathcal{O}(1) = \mathcal{O}(A)$.

Under the assumptions of Definition 3, the graded ring

$$R(X) = R\Big(X, \bigoplus \mathcal{O}_X(i)\Big) = \bigoplus_{i\geq 0} H^0(X, \mathcal{O}_X(i)) \tag{8}$$

is a finitely generated $k$-algebra, of the form $R(X) = k[x_0, \dots, x_N]/I_X$ with weighted generators $x_i \in H^0(X, \mathcal{O}_X(w_i))$ and a weighted homogeneous ideal $I_X$. The affine variety $\mathcal{C}_X = \operatorname{Spec} X$ is the *weighted cone* over $X$; the grading induces an action of the multiplicative group $\mathbb{G}_m$ on $R(X)$ and $\mathcal{C}_X$ that defines the quotient $X = \operatorname{Proj} R(X) = (\mathcal{C}_X \setminus 0)/\mathbb{G}_m$. Under my assumptions, $\mathcal{C}_X$ is nonsingular outside the origin, and the orbifold behaviour of $X, \bigoplus \mathcal{O}_X(i)$ comes from isolated orbits with cyclic isotropy subgroups $\mu_r \subset \mathbb{G}_m$. The generators $x_i$ give $\mathcal{C}_X \subset \mathbb{A}^{N+1}$ and $X \subset \mathbb{P}(w_0, \dots, w_N)$, where $\mathbb{A}^{N+1}$ is affine space with coordinates $x_0, \dots, x_N$ and $\mathbb{P}^N(w_0, \dots, w_N)$ is weighted projective space (wps or w$\mathbb{P}^N$) with homogeneous coordinates $x_i$ of weight $w_i$.

**Definition 4** Write $P_i(R) = \dim_k R_i$ for the dimension of the $i$th graded piece of a finitely generated graded ring $R = \bigoplus_{i\geq 0} R_i$; by abuse, I call $P_i(X) = P_i(R(X)) = h^0(X, \mathcal{O}(i))$ the $i$th *plurigenus* of $X$. The *Hilbert series* of $R(X)$ or of $X$ is the formal power series

$$P_X(t) = P_{R(X)}(t) = \sum_{i\geq 0} P_i(R) t^i. \tag{9}$$

It is known to be a rational function $\operatorname{Num}(t)/\prod(1 - t^{w_j})$ with denominator corresponding to the generators of $R$. The main point of this paper is that the generating function $P_X(t)$ is often simpler than the individual $P_i(X)$. My problem is to calculate $P_X(t)$ under extra conditions.

**Definition 5** I say that $X, \bigoplus \mathcal{O}_X(i)$ is *projectively Gorenstein* if $R(X)$ is a Gorenstein graded ring. This is equivalent (compare [GW] and [W]) to the following cohomological conditions:

- $H^j(X, \mathcal{O}_X(i)) = 0$ for all $j$ with $0 < j < \dim X$ and all $i$;

- the $i$th graded piece $R_i$ of $R(X)$ equals the complete linear system $H^0(X, \mathcal{O}_X(i))$ (a projective normality assumption, already implicit in the definition of $R(X)$);
- the orbifold canonical sheaf of $X$ is of the form $\omega_X = \mathcal{O}_X(k_X)$ for some integer $k_X$, the *canonical weight* of $X$.

The final condition is stated here for $n \geq 2$, when $\omega_X$ is a divisorial sheaf. See 1.3 for orbifold behavior in codimension 1.

**Definition 6 (Mukai [Mu])** Let $X$ be a projectively Gorenstein simply polarised orbifold with isolated orbifold points as above. The *coindex* of $X$ is defined by $c = k_X + n + 1$ where $k_X$ is as in Definition 5 and $n = \dim X$.

**Remark 7** The coindex is invariant under passing to a hyperplane section (of weighted degree 1), since the canonical class increases by 1 by adjunction, while the dimension decreases by 1. Mukai's definition is well known for nonsingular projectively Gorenstein varieties when $c \geq 0$. Clearly,

- $\mathbb{P}^n$ has coindex 0;
- a quadric hypersurface $Q \subset \mathbb{P}^{n+1}$ has coindex 1;
- a normal elliptic curve, del Pezzo surface or Fano 3-fold of index 2 has coindex 2;
- a canonical curve, K3 surface or anticanonical Fano 3-fold has coindex 3;
- a surface of general type or Calabi–Yau 3-fold has coindex 4.

See Remark 9 for their Hilbert series.

## 1.2 Model theorem

**Theorem 8** *A nonsingular projectively Gorenstein variety $X$ has Hilbert series*

$$P_X(t) = \frac{\mathrm{Num}(t)}{(1-t)^{n+1}}, \tag{10}$$

*where* $\mathrm{Num}(t)$ *is a symmetric polynomial of degree* $c = \operatorname{coindex} X$.

**Proof** This is an elementary consequence of Hirzebruch Riemann–Roch, plus the vanishing assumption in Definition 5 and Serre duality.

In detail, RR implies that $\chi(\mathcal{O}_X(i))$ is a polynomial of degree $n$ in $i$:

$$\chi(\mathcal{O}_X(i)) = \chi(\mathcal{O}_X(iA)) = \int \mathrm{ch}(iA) \cdot \mathrm{Td}(X) \tag{11}$$

where $A = c_1(\mathcal{O}_X(1))$. By the vanishing assumption there is no intermediate cohomology, and by Serre duality $h^n(\mathcal{O}(i)) = h^0(\mathcal{O}(k_X - i))$, so that

$$\begin{aligned} \chi(\mathcal{O}_X(i)) &= h^0(X, \mathcal{O}(i)) + (-1)^n h^n(X, \mathcal{O}(i)) \\ &= P_i(X) + (-1)^n P_{k_X - i}(X). \end{aligned} \tag{12}$$

In particular $P_i(X)$ is a polynomial in $i$ of degree $n$ for $i \geq k_X + 1$. (Note that when $k_X < 0$, this says that $H^0(\mathcal{O}_X(i)) = H^n(\mathcal{O}_X(i)) = 0$ for all intermediate values $k_X + 1 < i < 0$; this is the reason a nonsingular Fano $n$-fold has Fano index $-k_X \leq n + 1$.)

It follows that $(1-t)^{n+1} P_X(t)$ is a polynomial of degree $k_X + n + 1 = c$. Indeed, when I multiply $P_X(t)$ by $(1-t)$, the $i$th coefficient of the product $(1-t)P_X(t)$ is $P_i(X) - P_{i-1}(X)$; now differencing a polynomial of degree $n$ gives a polynomial of degree $n-1$, and therefore $P_i(X) - P_{i-1}(X)$ equals a polynomial of degree $n-1$ in $i$ for $i \geq k_X + 2$, and so on by induction.

Now Serre duality implies that $P_X(t)$ satisfies the functional equation

$$t^{k_X} P(1/t) = (-1)^{n+1} P(t) \tag{13}$$

under $t \mapsto 1/t$. To prove this, consider the formal sum $\sum_{i \in \mathbb{Z}} \chi(\mathcal{O}_X(i)) t^i$. Since by Serre duality

$$\chi(\mathcal{O}_X(i)) = P_i(X) + (-1)^n P_{k_X - i}(X) \tag{14}$$

with $P_i(X) = 0$ for $i < 0$ and $P_{k_X - i}(X) = 0$ for $i > k_X$, it makes sense to divide the formal sum up as a formal power series $P_X(t)$ in positive powers of $t$ plus $(-1)^n t^{k_X} P_X(1/t)$, which is $t^{k_X}$ times a power series in negative powers of $t$. On the other hand, multiplying the formal sum formally by $(1-t)^{n+1}$ gives zero, since $\chi(\mathcal{O}_X(i))$ is a polynomial of degree $n$ in $i$. This proves (13).

(13) implies that the numerator of $P(X)$ is a symmetric polynomial, and completes the proof. QED

**Remark 9** For low values of $c$, the numerator $\mathrm{Num}(t)$ in (10) is

$$\begin{array}{|c|l|}\hline \text{coindex} & \mathrm{Num}(t) \\ \hline\hline c=0 & 1 \\ c=1 & 1+t \\ c=2 & 1+dt+t^2 \\ c=3 & 1+(g-2)t+(g-2)t^2+t^3 \\ c=4 & 1+at+bt^2+at^3+t^4 \\ c=5 & 1+at+bt^2+bt^3+at^4+t^5 \\ \hline \end{array} \tag{15}$$

This is the Hilbert series converse of Remark 7. The form of these polynomials is familiar: for example, a regular surface of general type has $c = 4$, with $a = p_g - 3$ and $b = K^2 - (2p_g - 4)$. The formula itself works perfectly well even if $p_g = 0$, so $a = -3$. The sum of the coefficients, that is $\mathrm{Num}(1)$, equals the degree of the polarised variety $X$.

My convention is to take the first $[c/2]$ coefficients of $\mathrm{Num}(t)$ as the basic global invariants of $X$. One effect is that we study Hilbert series in terms of plurigenera themselves; relating the initial plurigenera to the topological invariants of $X$ (the Todd classes, the terms in $\int \mathrm{ch}(iA)\,\mathrm{Td}(X)$ of Hirzebruch RR) becomes a secondary issue.

The classic case is when $R(X)$ has a regular sequence $x_0, \dots, x_n$ in degree 1; geometrically, this means that $|\mathcal{O}_X(1)|$ is a free linear system. Then $R(X)$ is a free graded module over the polynomial ring $k[x_0, \dots, x_n]$, and its generators map one-to-one to a $k$-vector space basis of the Artinian quotient ring $R(X)/(x_0, \dots, x_n)$; the Hilbert numerator of $R(X)$ is the Hilbert series of this Artinian quotient. Passing to the numerator of $P_X(t) = \frac{\mathrm{Num}(X)}{(1-t)^{n+1}}$ has the effect of normalising $X$ to dimension $-1$.

## 1.3 The curve case

The statement of Theorem 1 does not cover the elementary case of a curve $C$ with isolated orbifold points: the simple device used throughout 1.1 of avoiding the full graded structure sheaf of $X, \mathcal{O}_X(i)$ by writing $\mathcal{O}_X(i) = \mathcal{O}_X(iA)$ with $A$ a $\mathbb{Q}$-Cartier Weil divisor does not work: it is not true that the sheaf $\mathcal{O}_X(i)$ is determined by $\mathcal{O}_X(1)$, because the multiplication maps $\mathcal{O}_C(i) \otimes \mathcal{O}_C(j) \to \mathcal{O}_C(i+j)$ are no longer isomorphisms in codimension 1.

The way around this is simple and well known.[1]

[1] Other drafts of the same remark:

**Recipe 11** Replace $A$ by a $\mathbb{Q}$-divisor that includes the fractional term $\frac{b}{r}P$ for each orbifold point $P$ of type $\frac{1}{r}(a)$ on $C$, where $b$ is the inverse of $a$ modulo $r$; that is, write

$$A = A_0 + \sum_{\mathcal{B}} \frac{b}{r} P \quad \text{with } A_0 \text{ an integral divisor.}$$

At the same time, replace $K_C$ by $K_{C,\mathrm{orb}} = K_C + \sum_{\mathcal{B}} \frac{r-1}{r} P$.

Then $\mathcal{O}_X(i) = \mathcal{O}_X(iA) := \mathcal{O}_X([iA])$, Serre duality takes the correct form, and the graded ring $R(C, A)$ is Gorenstein if and only if $K_{C,\mathrm{orb}} = k_C A$ for some integer $k_C$.

**Discussion** I explain why the construction is right, leaving the details to you (see Demazure [De] and Watanabe [W]); this also appears subliminally in many places in papers by Kawamata, Reid, Shokurov, and others.

The assumption on $C$ is that at each orbifold point, the local parameter $z_P$ of the complex curve $C$ is $z_P = w_P^r$, where $w_P$ is the *orbinate* (orbifold coordinate, that is, the coordinate on the overlying orbifold cover), and the $\mu_r$ action is $w_P \mapsto \varepsilon^a w_P$. The sheaf $\mathcal{O}_C(i)$ consists locally of the $i$th eigensheaf

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Orbifold behaviour in codimension 1 and 0 is natural and simple, and can't be avoided if we want a treatment of orbifolds that includes induction by restriction to smaller strata. The point can be viewed in terms of stacks: the space is only the underlying space $|X|$ (or coarse moduli space) for the bigger structure $X, \bigoplus \mathcal{O}(i)$. In particular, a general orbifold has a graded dualising sheaf $\omega_X(i) = \mathcal{O}_X(k_X - i)$.

The material of 1.1 uses the traditional "well-formed" device of higher dimensional geometry that allows one to avoid mentioning stacks when the orbifold behaviour is in codimension $\geq 2$: work with a Weil $\mathbb{Q}$-Cartier divisor $A$ and set $\mathcal{O}_X(i) = \mathcal{O}_X(iA)$.

**Remark 10 (Orbifold in codim 0 and 1)** When $n = 1$, the orbifold points in codimension 1 require extra care: the notation $\mathcal{O}(i) = \mathcal{O}_X(iA)$ only makes sense after introducing a $\mathbb{Q}$-Weil divisor $A = \sum \frac{a_j}{r_j} E_j$, as in Demazure [De], and $\mathcal{O}(1)$ does not determine $\mathcal{O}(i)$. Orbifold curves are pretty simple (see Section 6); however, to handle orbifold behaviour in codimension 1 the graded structure sheaf $\bigoplus \mathcal{O}(i)$ must be specified. There are in any case theoretical advantages in thinking of $\bigoplus \mathcal{O}(i)$ systematically as a graded structure sheaf.

The final condition is stated here for $n \geq 2$, when $\omega_X$ is a divisorial sheaf. More care is needed to handle orbifold behavior in codimension 1: namely, rather than a single dualising sheaf $\omega_X$, we need the graded dualising sheaf of the graded structure sheaf $\bigoplus \mathcal{O}_X(i)$. The case $n = 1$ (orbifold curves) can be treated in terms of fractional divisors $A = \sum \frac{a_i}{r_i} P_i$ and the orbifold canonical class $K_{C,\mathrm{orb}} = K_C + \sum \frac{r_i - 1}{r_i} P_i$; see 1.3, [De] and [W].

of this action; that is, it consists of monomials $w_P^m$ with $m \geq 0$ and $am \equiv i \mod r$, that is $m \equiv bi \mod r$, together with power series consisting of sums of these (formal, convergent or algebraic power series, according to taste). Therefore the sheaf $\mathcal{O}_C(i)$ is based by $w_P$ to the $[\frac{bi}{r}]$ power. In the same way, the orbifold canonical class is based by $\mathrm{d}w_P$, which is equal to a constant times $w_P^{\frac{r-1}{r}}\,\mathrm{d}z_P$.

**Exercise 12** If $C$ is an orbifold curve with a basket $\mathcal{B} = \{P, 1/r(a)\}$, and $\bigoplus \mathcal{O}_C(i)$ is represented by $\mathcal{O}(iA)$ as just discussed, then $R(C, \bigoplus \mathcal{O}_C(i))$ is Gorenstein with index $k_C$ if and only if $a + k_C \equiv 0 \mod r$ for each $P \in \mathcal{B}$, and then its Hilbert series is given by

$$P_C(t) = P_I + \sum_{\mathcal{B}} P_{\mathrm{orb}}(\tfrac{1}{r}(a), k_C) \tag{16}$$

where as in Theorem 1, the initial term $P_I$ is $\frac{1}{(1-t)^2}$ times a Gorenstein symmetric polynomial of degree $k_C + 2$. Each orbifold term is obtained by calculating $b$, the inverse of $a$ mod $r$, and taking $\frac{1-t^b}{1-t}$ modulo $\frac{1-t^r}{1-t}$ written out as a Laurent polynomial supported in the appropriate interval.

**Example 13** Let $C = \mathbb{P}^1$ and take $A = \frac{3}{5}P + \frac{1}{2}Q - R$ (where $P, Q, R \in C$ are distinct points); then $K_{C,\mathrm{orb}} = \frac{4}{5}P + \frac{1}{2}Q - 2R$, so $-7A \sim K_{C,\mathrm{orb}}$. One sees that $R(C, A) = k[x, y]$ where $\mathrm{wt}\, x, y = 2, 5$. In degree 2, $x$ vanishes at $P$ to order $\frac{1}{5}$, so $P = (0, 1)$ is a $\frac{1}{5}(2)$ orbifold point with $x$ as orbinate. The case of $\mathbb{P}^1(a, b)$ with any coprime $a, b$ is similar.

**Example 14** The weighted projective line $X = \mathbb{P}(2, 5)$ has orbifold points of type $\frac{1}{5}(2)$ and $\frac{1}{2}(5) = \frac{1}{2}(1)$, in the global context $k_X = -7$. The initial term $P_I = 0$ whenever $c < 0$. The orbifold terms are

$$\begin{aligned} P_{\mathrm{per}}(\tfrac{1}{5}(2)) &= \frac{1}{1-t^5} \times \frac{1}{5} \times (-3t - t^2 - 4t^3 - 2t^4), \\ P_{\mathrm{tot}}(\tfrac{1}{5}(2), -7) &= P_{\mathrm{per}} + \frac{-t^{-2} + t^{-1} - (2/5)t}{(1-t)^2} = \frac{-t^{-2} - t}{(1-t)(1-t^5)}, \end{aligned} \tag{17}$$

and

$$\begin{aligned} P_{\mathrm{per}}(\tfrac{1}{2}(1)) &= \frac{1}{1-t^2} \times \frac{1}{2} \times (-t), \\ P_{\mathrm{tot}}(\tfrac{1}{2}(1), -7) &= P_{\mathrm{per}} + \frac{t^{-2} - t^{-1} + 1 - (1/2)t}{(1-t)^2} = \frac{t^{-2}}{(1-t)(1-t^2)}. \end{aligned} \tag{18}$$

Adding these gives

$$\frac{-t^{-2}-t}{(1-t)(1-t^5)} + \frac{t^{-2}}{(1-t)(1-t^2)} = \frac{1}{(1-t^2)(1-t^5)} \,. \tag{19}$$

Although mystifying at first sight, these calculations are really very easy, and understanding them illuminates the general case. In (17), the terms $\frac{1}{1-t^5}(-\frac{3}{5}t - \frac{1}{5}t^2 - \frac{4}{5}t^3 - \frac{2}{5}t^4)$ are the periodically repeating fractional parts lost on rounding down $\mathcal{O}_C(\frac{3i}{5}P)$. Since the plurigenera are integers, I must add some global term to compensate (thus adding to the degree in the polynomial part of RR); adding $\frac{3}{5}t$ in the numerator of $P_{\mathrm{grow}}$ would give

$$\begin{aligned}\frac{1}{1-t^5}\left(-\frac{3}{5}t - \frac{1}{5}t^2 - \frac{4}{5}t^3 - \frac{2}{5}t^4\right) + \frac{3}{5} \times \frac{t}{(1-t)^2}& \\ = t^2 + t^3 + 2t^4 + 3t^5 + 3t^6 + 4t^7 + 4t^8 + 5t^9 + 6t^{10} + \cdots& \\ = \frac{t^2+t^4+t^5}{(1-t)(1-t^5)}\,,&\end{aligned} \tag{20}$$

the natural integral growth of $[\frac{3i}{5}]$, incrementing when $i \equiv 2, 4$ or $0 \bmod 5$. However, in the context $k = -7$, I want the contribution to be symmetric of degree $-7$ in the sense of the functional equation (13). So rather than add $\frac{3}{5}t$ in the numerator I add $-t^{-2} + t^{-1} - \frac{2}{5}t$; the numerator $-t^{-2} - t$ in (17) is symmetric of degree $-1$, so that the whole contribution $P_{\mathrm{tot}}$ has degree $-7$.

In (18) the numerator $t^{-2}$ is symmetric of degree $-4$ (because the only term $t^{-2}$ is the centre of the symmetry), so $P_{\mathrm{tot}}$ again has degree $-7$.

## 1.4 Isolated orbifold singularities

Now let $X$ be projectively Gorenstein with isolated orbifold singularities, with $k_X$ and $c$ as in Definitions 5–6. The ingredients in the plurigenus formula are as follows:

- the dimension $n$;
- the canonical weight $k_X$ (see Definition 5);
- the coindex $c = k_X + n + 1$;
- the first $[c/2]$ plurigenera $P_i(X)$ for $i = 1, \ldots, [c/2]$;

- a basket $\mathcal{B}$ of isolated orbifold points $\mathcal{B} = \{\frac{1}{r}(a_1, \dots, a_n)\}$, with $r \geq 2$.

  For an orbifold point $\frac{1}{r}(a_1, \dots, a_n)$, the isolated assumption is that each $a_1, \dots, a_n \in [1, r-1]$ is coprime to $r$.

  My assumption that $X$ is projectively Gorenstein with $\omega_X = \mathcal{O}_X(k_X)$ implies that each $\frac{1}{r}(a_1, \dots, a_n) \in \mathcal{B}$ satisfies

$$k_X + \sum_{j=1}^{n} a_j \equiv 0 \mod r. \tag{21}$$

I now use these ingredients to cook up an *initial term* $P_{I,X}(t)$, and for each point $\frac{1}{r}(a_1, \dots, a_n) \in \mathcal{B}$ an *orbifold contribution* $P_{\text{orb}}(\frac{1}{r}(a_1, \dots, a_n), k_X)$, each computed by a simple recipe, so that

$$P_X(t) = P_{I,X}(t) + \sum_{\mathcal{B}} P_{\text{orb}}\big(\tfrac{1}{r}(a_1, \dots, a_n), k_X\big). \tag{22}$$

Note that "initial term" certainly does not mean "leading term": it does the initial plurigenera, but not the leading order of growth.

**Definition 15 (Initial term)** The *initial term* $P_{I,X}(t)$ is

$$P_{I,X}(t) = \frac{A(t)}{(1-t)^{n+1}}\,, \tag{23}$$

where $A(t)$ is a symmetric polynomial of degree $c$ with integer coefficients, uniquely determined by the condition that the formal power series $P_{I,X}$ has the given $P_i(X)$ as coefficient of $t^i$ up to $i = [c/2]$ (with $P_0 = 1$ if $c \geq 0$).

If $c < 0$ then also $[c/2] < 0$, and $P_I = 0$.

**Recipe 16** (1) Set $A_0 = \sum_{i=0}^{[c/2]} P_i t^i$ (this is 0 is $c < 0$);

(2) set $A_1 = (1-t)^{n+1} A_0$ and $P'_i =$ coefficient of $t^i$ in $A_1$ for $i = 0, \dots, [c/2]$;

(3) finally, set $A(t) = \sum_{i=0}^{c} P''_i t^i$ where $P''_i = P'_i$ or $P'_{c-i}$.

**Example 17** Take $n = 3$, $c = 5$, $P_1 = 3$, $P_2 = 7$, $A_0 = 1 + 3t + 7t^2$; then $A_1 = (1-t)^4 A_0 = 1 - t + t^2 + \cdots$, so that $P_I(t) : \frac{1-t+t^2+t^3-t^4+t^5}{(1-t)^4}$; you check that $P_I(t) = \frac{1+2t^3+t^6}{(1-t)^3(1-t^2)}$ is the Hilbert series of the nonsingular canonical 3-fold $X(6,6) \subset \mathbb{P}(1,1,1,2,3,3)$.

**Definition 18 (Orbifold term)** Let $\frac{1}{r}(a_1,\dots,a_n)$ and $k_X$ be as above. Its orbifold contribution is defined by

$$P_{\mathrm{orb}}\bigl(\tfrac{1}{r}(a_1,\dots,a_n),k_X\bigr)=\frac{B(t)}{(1-t)^n(1-t^r)},\tag{24}$$

where the numerator $B(t)$ is

- the inverse modulo $\dfrac{1-t^r}{1-t}=1+t+\cdots+t^{r-1}$ of $\prod\left(\dfrac{1-t^{a_i}}{1-t}\right)$
- as a Laurent polynomial with support in $[\gamma+1,\gamma+r-1]$, where $\gamma=[c/2]$.

**Magma function 19**

```
 function Qorb(r,LL,k) L :=
[ Integers() | i : i in LL ]; // this allows empty list if (k + &+L) mod
r ne 0 then error "Error: Canonical weight not compatible"; end if; n :=
#LL; Pi := &*[ R | 1-t^i : i in LL]; h := Degree(GCD(1-t^r, Pi)); //
degree of GCD(A,B) // -- simpler calc? l := Floor((k+n+1)/2+h); // If l
< 0 we need a kludge to avoid programming // Laurent polynomials
properly de := Maximum(0,Ceiling(-l/r)); m := l + de*r; A := (1-t^r) div
(1-t); B := Pi div (1-t)^n; H,al_throwaway,be:=XGCD(A,t^m*B); return
t^m*be/(H*(1-t)^n*(1-t^r)*t^(de*r)); end function;
```

Given $\frac{1}{r}(a_1,\dots,a_n)$, in the context of $K_X=kA$, the calculation is the Euclidean algorithm for the hcf of

$$A:=\frac{t^r-1}{t-1}=1+t+\cdots+t^{r-1},\quad\text{and}\quad B:=\prod\frac{t^{a_i}-1}{t-1}.\tag{25}$$

Write $h=\mathrm{hcf}(A,B)$ (which is 1 in the current case, with the $a_i$ coprime to $r$) and $l=[(k+n+1)/2]+\deg h$. (The $l$ just translates the support of the Laurent polynomial.) Now calculate the hcf by the Euclidean algorithm in the form

$$\mathrm{hcf}(A,t^lB)=:H=\alpha A+\beta t^lB,\tag{26}$$

and return $\frac{\beta t^l}{H(1-t)^n(1-t^r)}$. Since $\beta$ is in the range $[0,r-2]$ it follows that $\beta+l$ is in the range `[Floor((k+n+1)/2+h),Floor((k+n+1)/2+h)+r-2]` , as required.

## 1.5 Some progress on Hilbert series of CY orbifolds

The general formula is

$$P_I + \sum_{B\in\mathcal{B}} P_{\text{orb}}(B,0) + \sum_{C in \mathcal{C}} P_C(C)$$

here (1) the *initial term* is

$$P_I = \frac{1+at+bt^2+at^3+t^4}{(1-t)^4}.$$

In the point sum, each $B = \frac{1}{r}(a_1,a_2,a_3)$, and the term is

$$P_{\text{orb}}(B,0) = \frac{\text{Num}}{(1-t)^3/(1-t^r)}$$

where, in the isolated case, Num is the unique polynomial with support in $[3,r]$ which is the inverse of $\prod_{i=1}^n \frac{1-t^{a_i}}{1-t}$; in general you also have to take out $1/(1-t^b)$ for common factors, etc., and the function is given by Magma function Qorb of (19).

In the curve sum, each $C$ is of the form $1/r(a,r-a)$ plus extra data, and the term is

$$P_C = \frac{\text{Num}}{(1-t)^2(1-t^r)^2} = \frac{\text{Num}}{[1,1,r,r]}$$

where Num is a symmetric polynomial of $2r+2$ supported in $[3,2r-1]$. Num has $r-1$ arbitrary coefficients, and the initial expectation is that all values in an open range will occur.

**Remark 20 (Preliminary notes to myself)** I think I have progress on the 1-dim orbifold locus contributions, Namely, if $X$ has transverse $\frac{1}{r}(L)$ singularities along a curve $\Gamma$, in the context of $K_X = \mathcal{O}_X(k)$, the contribution is

$$A \times P_{\text{orb}}(\frac{1}{r}(L), k+r) \times \frac{1}{(1-t^r)} + B \times \frac{t^a}{(1-t)^n(1-t^r)} \tag{27}$$

where $A$ and $B$ are Gorenstein symmetric Laurent polynomials of given degree and support. I'm not quite sure, but one prediction is that "most" $A$ and $B$ within some range occur; but maybe there are divisibility or congruence conditions, or at the other extreme, only one or two $A$ and $B$ allowed. Each of $A$ and $B$ has approx $(r-1)/2$ free coefficients, which is reasonable since they contain implicitly the RR data for $\Gamma, \mathcal{O}_\Gamma(i)$, the normal bundle to $\Gamma$ (with its $\mathbb{Z}/r$ eigendecomposition) and all their twists. In any case, any choice of $A$ and $B$ give rise to Hilbert series with the right symmetry, so we can make 10 billion baskets of CYs in the very near future.

Let's try to say that more precisely. $P_{\mathrm{orb}}(\frac{1}{r}(L), k)$ is the contribution of a single point $\frac{1}{r}(L)$ on an $m$-fold, not necessarily isolated, where $m = \#L$. It has denominator $(1-t)^m(1-t^r)$ and is Gorenstein symmetric of degree $k$, and its numerator $\alpha$ is a Laurent polynomial of support of length $< r$, uniquely determined by the condition that

$$h = \mathrm{hcf}(F, G) = \alpha F + \beta G, \quad \text{where} \quad F = \prod_{a \in L}(1-t^a) \quad \text{and} \quad G = 1 - t^r. \tag{28}$$

The Magma function below says it all (and is to some extent tried and tested).

Now in (27), $A$ is Gorenstein symmetric of degree 0 and has support in $[(-r+1)/2, \dots, (r-1)/2]$. (e.g., for $r = 13$, something like

$$A = (1 - t^4 + t^5 - t^6 + t^{10})/t^5 \tag{29}$$

is allowed.) $B$ is Gorenstein symmetric with

$$\deg B = k - 2a + n + r \tag{30}$$

(so that the whole term in (27) has degree $k$), and has support in

$$[\deg B + (-r+1)/2, \deg B + (r-1)/2]. \tag{31}$$

I still have to test this against the famous 7555 hypersurfaces to find out how many $A$ and $B$ to expect.

# 2 Periodic term

A more straightforward approach to isolated orbifold points is to think of a contribution that is periodic with period $r$, with the whole Porb made up of Pper plus an initial term Pgrow.

**Definition 21 (Periodic term)** The *periodic term* of an isolated orbifold point $\frac{1}{r}(a_1, \dots, a_n)$ is the rational function

$$P_{\mathrm{per}}(\tfrac{1}{r}(a_1, \dots, a_n)) = \frac{N_{\mathrm{per}}}{1 - t^r}, \tag{32}$$

where the numerator is the inverse modulo $1 + t + \cdots + t^{r-1}$ of $\prod(1 - t^{a_i})$ written out as

$$N_{\mathrm{per}}(\tfrac{1}{r}(a_1, \dots, a_n)) = \sum_{i=1}^{r-1} b_i t^i \quad \text{with } b_i \in \mathbb{Q}. \tag{33}$$

**Example 22** Exercise: $P_{\mathrm{per}}(\frac{1}{5}(1,2)) = \frac{1}{1-t^5}(-\frac{1}{5}t - \frac{2}{5}t^3 - \frac{2}{5}t^4)$. [Hint: use the cyclotomic identity $\prod_{i=1}^{p-1}(1 - \varepsilon^i) = p$ to calculate $1/((1-\varepsilon)(1-\varepsilon^2))$ where $\varepsilon$ is a primitive 5th root of unity.]

See Lemma 25 for several more general recipes to calculate $P_{\mathrm{per}}$, for the $b_i$ as Dedekind sums, and for the Serre duality symmetry between them. This part of the Hilbert series is rational and periodic, in particular bounded: the denominator $1 - t^r$ just makes the terms from 1 to $r-1$ repeat with period $r$. The periodic part $P_{\mathrm{per}}(\frac{1}{r}(a_1, \dots, a_n))$ records the deviation of $P_i(X)$ from being a polynomial in $i$.

**Growing term** The growing term depends on $\frac{1}{r}(a_1, \dots, a_n)$ and on the global canonical weight $k_X$ (equivalently, the coindex $c = k_X + n + 1$). It is

$$P_{\mathrm{grow}}(\tfrac{1}{r}(a_1, \dots, a_n), k) = \frac{B(t)}{(1-t)^{n+1}}\,, \tag{34}$$

where $B \in \mathbb{Q}[t, t^{-1}]$ is a Laurent polynomial uniquely determined by the condition that

$$N_{\mathrm{per}}(1-t)^n + B(1 + t + \cdots + t^{r-1}) \tag{35}$$

is a rational linear combination of $t^i$ with $[c/2] < i < [c/2] + r$ (a Laurent polynomial supported in $\big[[c/2]+1, [c/2]+r-1\big]$). See Lemma 23 for a proof and a recipe.

# 3 Some lemmas

I calculate the contributions

$$P_{\mathrm{tot}}(\tfrac{1}{r}(a_1, \dots, a_n), k) = P_{\mathrm{per}}(\tfrac{1}{r}(a_1, \dots, a_n)) + P_{\mathrm{grow}}(\tfrac{1}{r}(a_1, \dots, a_n), k) \tag{36}$$

for an isolated quotient singularity $\frac{1}{r}(a_1, \dots, a_n)$ on a polarised variety with $K = kA$.

**Lemma 23** *Consider a polynomial*

$$B = \sum_{i=1}^{r-1} b_i t^i \in \mathbb{Q}[t], \tag{37}$$

*and suppose given* $r, n \in \mathbb{N}$ *and an interval* $J = [d+1, \dots, d+r-1]$ *of* $r-1$ *consecutive integers. Then there exists a unique Laurent polynomial*

$$A = \sum_{j \in J} \alpha_j t^j \in \mathbb{Q}[t, t^{-1}] \tag{38}$$

*supported in* $J$ *such that* $A - (1-t)^n B$ *is divisible by* $1 + t + \cdots + t^{r-1}$*:*

$$A - (1-t)^n B = \frac{1-t^r}{1-t} L \tag{39}$$

*with* $L$ *a Laurent polynomial.*

It follows that

$$\frac{B}{1-t^r} + \frac{L}{(1-t)^{n+1}} = \frac{A}{(1-t)^n(1-t^r)}. \tag{40}$$

Later I write $P_{\text{per}} = \frac{B}{1-t^r}$, $P_{\text{grow}} = \frac{L}{(1-t)^{n+1}}$, so that $P_{\text{tot}} = P_{\text{per}} + P_{\text{grow}} = \frac{A}{(1-t)^n(1-t^r)}$.

**Proof** The quotient ring

$$V = \mathbb{Q}[t]/(1 + t + \cdots + t^{r-1}) \tag{41}$$

is an $(r-1)$-dimensional vector space based by $1, t, \dots, t^{r-2}$. However, $t$ maps to an invertible element of $V$, so that also

$$V = \mathbb{Q}[t, t^{-1}]/(1 + t + \cdots + t^{r-1}), \tag{42}$$

and the $r-1$ elements $t^j$ for $j \in J$ form another basis of $V$.

Therefore the class of $(1-t)^n B$ modulo the ideal of $\mathbb{Q}[t, t^{-1}]$ generated by $1 + t + \cdots + t^{r-1}$ can be written in a unique way as a linear combination of $t^j$ for $j \in J$. QED

**Definition 24** The numerator

$$N_{\mathrm{per}}(\tfrac{1}{r}(a_1, \dots, a_n)) = \sum_{i=1}^{r-1} b_i t^i \tag{43}$$

of $P_{\mathrm{per}}$ is defined as the inverse modulo $1 + t + \cdots + t^{r-1}$ of $\prod_{j=1}^{n}(1 - t^{a_j})$, and $P_{\mathrm{per}}$ itself is defined by

$$P_{\mathrm{per}} = \frac{N_{\mathrm{per}}}{1 - t^r}. \tag{44}$$

**Lemma 25** *(1) When $n = 1$,*

$$N_{\mathrm{per}}(\tfrac{1}{r}(a)) = \frac{1}{r} \sum_{i=1}^{r-1} -\overline{bi}\, t^i, \tag{45}$$

*where as usual*

*$b$ is the inverse of $a$ modulo $r$, and*
*in $\overline{bi}$, the bar denotes smallest residue mod $r$.*

*(2) For general $n$,*

$$N_{\mathrm{per}}(\tfrac{1}{r}(a_1, \dots, a_n)) = \prod_{j=1}^{n} N_{\mathrm{per}}(\tfrac{1}{r}(a_j)) \mod 1 + t + \cdots + t^{r-1}. \tag{46}$$

**Proof of (1)** Consider

$$(1 - t^a) \left( \sum_{i=1}^{r-1} -\overline{bi}\, t^i \right) \sum_{i=1}^{r-1} -\overline{bi}\, t^i + \sum_{i=1}^{r-1} \overline{bi}\, t^{a+i} \tag{47}$$

Working modulo $1 + t + \cdots + t^{r-1}$ allows me to substitute $t^r = 1$, and to subtract a scalar multiple of $1 + t + \cdots + t^{r-1}$ or of $t + t^2 + \cdots + t^r$. On applying these rules, the right-hand side evaluates to $r$.

For example, $r = 7$ and $a = 2$; then $b = 4$ and we consider

$$-4t - t^2 - 5t^3 - 2t^4 - 6t^5 - 3t^6; \tag{48}$$

multiplying by $1-t^2$ gives

$$\begin{array}{cccccccc} -4t & -t^2 & -5t^3 & -2t^4 & -6t^5 & -3t^6 & & \\ & & +4t^3 & +t^4 & +5t^5 & +2t^6 & +6t^7 & +3t^8; \end{array} \tag{49}$$

according to my rules, I can replace $3t^8$ by $3t$ and $6t^7$ by $6 = (7-1)$, and the whole sums to

$$7 - (1 + t + t^2 + \cdots + t^6) = 7. \tag{50}$$

In the general case, for clarity, break up the second sum as a sum over $a+i \le r-1$ and another over $a+i \ge r$, and change the dummy index from $i$ to $j = a+i$. Then the second sum is

$$\sum_{j=a+1}^{r-1} \overline{b(j-a)}t^j + \sum_{j=r}^{r+a-1} \overline{b(j-a)}t^{j-r}$$
$$= \sum_{j=a+1}^{r-1} (\overline{bj}-1)t^j + (r-1)t^r + \sum_{j=r+1}^{r+a-1} (\overline{bj}-1)t^j. \tag{51}$$

In fact, the coefficient of $t^j$ is $\overline{bj}-1$ whenever $\overline{bj} \neq 0$. The only exception is the coefficient of $t^r$, which has $\overline{b(j-a)} = r-1$. Now in each term with $j \ge r$, substitute $t^r \mapsto 1$ to get (as in the example)

$$(1-t^a) \sum -\overline{bi}t^i = r - (1 + t + \cdots + t^{r-1}) = r. \tag{52}$$

(The missing term $i = a$ in the second sum has $\overline{ab} = 1$, so its coefficient is zero.) QED

# 4 Current status

An orbifold $X$ has strata $\{X_i\}_{i \in I}$ and open strata $X_i^\circ = X_i \setminus X_j$, each with a characteristic isotropy group; here because of the quasismooth simply polarised assumptions, the only isotropy groups are cyclic $\mu_{r_i} \subset \mathbb{C}^*$. Write $n = \dim X$, $d_i = \dim X_i$, and assume that $X$ is projectively Gorenstein with canonical degree $k_X$. I am gunning for a result of the following shape.

**Conjecture 26** *$P_X(t)$ is of the form $\sum P_i$, where the denominator of $P_i$ is $(1-t)^{n-d_i}(1-t^{r_i})^{d_i+1}$ and its numerator is an integral Gorenstein symmetric polynomial of given degree and support.*

*More precisely, the degree is such that the whole fraction $P_i$ is Gorenstein symmetric of degree $k_X$, and the support is an interval $I \subset \mathbb{Z}$ of length $(d_i+1)r_i - 1$, so that $\{t^i \bigm| i \in I\}$ bases $\mathbb{Z}[t,t^{-1}]/((1-t^{r_i})/(1-t))$.*

The problem is not so much to prove the conjecture, rather to understand the terms in it well enough so that we can predict them. At the moment, the interesting open question is the case of curve orbifold locuses, such as those on a normal 3-fold orbifold (for example, the 7555 qsmooth CY hypersurfaces), or an orbifold surface (for example, the $\mathbb{P}^2(a,b,c)$ with no factor common to all of $a,b,c$). In the CY case (see Buckley [B]), an orbifold curve $C$ of transverse type $1/r(a,r-a)$ on a CY 3-fold may pass through some nonisolated point singularities ("dissident points" in [B]), and the problem is to write the term $P_C$ in terms of the degree of $C$ and of its isotypical normal bundles, and its basket singularities.

There are two main methods of proceeding:

(1) Reverse engineer examples. There are huge numbers of examples for which most of the terms are known, and in particular cases one can deduce new orbifold terms from old ones.

(2) Massage the formulas of [B] into my preferred form. The information is practically all contained in [B], but not expressed in terms of integral polynomials with given symmetry degree and support, and without the nice interpretation in ice-cream terms.

# 5 Parsing curves on CY in Magma

This is a little suite of Magma functions that provide a routine to reverse engineer the plurigenus formula for each $Y = Y_d \subset \mathbb{P}(a_1,\dots,a_5)$ for $A = [a_1,\dots,a_5,d]$ in the list of 7555 qsmooth hypersurface. Here $P(A) = \frac{1-t^d}{\prod(1-t^{a_i})}$ and $P_I(A)$ are obvious things. `Bask(A)` is selfcontained; it just calculates the list `[ai mod r, d mod r]` for $A = [a_1,\dots,a_5,d]$, and puts the singular points and curves it finds into baskets B and C. PointTerms is the sum over point terms, using the preprogrammed function Qorb. $X(A)$ is the sum of the curve terms, treated here as an unknown to be taken apart. $X(A)$ is supposed to be a sum of terms `Num_r/Denom([1,1,r,r])` over the baskets. The function PC determines the different numerators `Num_r` using `PartialFractionDecomposition` plus a couple of little twists. It goes wrong if $Y$ has curves of singularities with index having a common factor (or maybe also in other cases) – this is recent and only tested in a few hundred cases.

See below for examples of its use.

```
 /* Given a qsmooth CY hypersurface Y(d) in PP(a1,.. a5)
in the form A := [a1,..a5,d], parse it into the form PI (initial term)
plus sum Porb(1/r(a,b,c)) with 1/r(a,b,c) in PointBasket plus sum
Ar/[1,1,r,r] where Ar is a symmetric polynomial of symmetric degree 2r+2
with support in [3..2r-1]. */

function P(A) return (1-t^A[6])/Denom(A[1..5]); end function;

function PI(A) // Initial term P_I, only for CY 3folds n1 := #[i : i in
A | i eq 1]; n2 := #[i : i in A | i eq 2]; return (1 + (n1-4)*t +
(n2+Binomial(n1-3,2))*t^2 + (n1-4)*t^3 + t^4)/(1-t)^4; end function;

function Bask(A); B := []; C := []; Relevant := [r : r in [2..A[5]] | r
eq GCD([Integers() | A[i] : i in [1..5] | A[i] mod r eq 0])]; for r in
Relevant do Amod := [a mod r : a in A ]; case [#[a : a in Amod[1..5] | a
eq 0],#[a : a in Amod[6..6] | a eq 0]]: // when 0:; // no sing, do
nothing when 4:; //error when 5:; //error when [1,0]: Append(~B,
Insert(Exclude(Exclude(Amod[1..5],Amod[6]),0),1,r)); // end this case:
If ai divides d then Pi not on X when [2,1]: // calculate the number
S^0(d) in PP(ai,aj) Num := Floor(A[6]/LCM([a : a in A[1..5] | a mod r eq
0])); for i in [1..Num] do Append(~B,
Insert(Exclude(Exclude(Amod[1..5],0),0),1,r)); end for; // end if; If r
does not divides d then Lij is line of 1/r when [2,0]: Append(~C,
Insert(Exclude(Exclude(Exclude(Amod[1..5],0),0),Amod[6]),1,r)); when
[3,1]:; // necessarily r divides d and curve of 1/r Append(~C,
Insert(Exclude(Exclude(Exclude(Amod[1..5],0),0),0),1,r)); end case; end
for; return B, C; end function;

function PointTerms(B) return &+[K | Qorb(b[1],b[2..4],0) : b in B]; end
function;

function X(A) return P(A)-PI(A)-PointTerms(Bask(A)); end function;

function PC(A) // The curve terms P_C B,C := Bask(A); YY :=
```

```
PartialFractionDecomposition(X(A)/t^3*(1-t)^4); return
[t^3/(1-t)^4*&+[K|y[3]/y[1]^y[2] : y in YY | IsDivisibleBy(1-t^r,y[1])]
where r is c[1] : c in C]; end function;

===================

for example, a little segment from the famous 7555:

AA:=[[1,12,27,32,36,108],[1,12,27,40,40,120],[1,12,27,68,96,204],[1,12,
27,68,108,216],[1,12,32,39,45,129],[1,12,33,40,46,132],[1,12,33,92,138,
276],[1,12,39,52,52,156],[1,12,39,52,65,169],[1,12,39,52,92,196],[1,12,
39,52,103,207],[1,12,39,104,156,312],[1,12,40,93,134,280],[1,12,41,96,
138,288],[1,12,42,98,141,294],[1,12,51,64,127,255],[1,12,51,64,128,256],
[1,12,53,120,186,372],[1,12,54,68,81,216],[1,12,54,122,189,378],[1,12,64
,153,230,460],[1,12,66,92,105,276],[1,12,66,158,237,474],[1,12,76,177,
266,532],[1,12,77,180,270,540],[1,12,78,104,117,312],[1,13,15,30,46,105]
,[1,13,21,28,62,125],[1,13,21,35,35,105],[1,13,21,35,69,139],[1,13,22,29
,52,117],[1,13,23,28,32,97],[1,13,28,56,98,196],[1,13,28,70,111,223],[1,
13,29,43,73,159],[1,13,29,44,74,161],[1,13,30,75,106,225],[1,13,34,41,88
,177],[1,13,34,61,108,217],[1,13,34,95,142,285],[1,13,41,96,150,301],[1,
13,41,109,163,327]];

> AA[34]; [ 1, 13, 28, 70, 111, 223 ] > Bask(AA[34]); [ [ 13, 1, 5, 7 ],
[ 28, 1, 13, 14 ], [ 70, 1, 28, 41 ], [ 111, 13, 28, 70 ] ] [ [ 14, 1,
13 ] ] > X(AA[34]);
(-t^22-t^20-t^18-t^17-t^16-t^15-t^14-t^13-t^12-t^10-t^8)
/(t^30-2*t^29+t^28-2*t^16+4*t^15-2*t^14+t^2-2*t+1) >
PartialFractionDecomposition(X(AA[34])/t^3*(1-t)^4); [ <t + 1, 2, 5/49>,
<t^6 - t^5 + t^4 - t^3 + t^2 - t + 1, 1, 29/196*t^4 - 87/196*t^3 +
125/196*t^2 - 47/98*t - 3/98>, <t^6 - t^5 + t^4 - t^3 + t^2 - t + 1, 2,
1/7*t^5 - 5/28*t^3 + 1/7*t^2 - 1/7*t + 5/28>, <t^6 + t^5 + t^4 + t^3 +
t^2 + t + 1, 1, -1/4*t^4 + 1/4*t^3 - 1/4*t^2>, <t^6 + t^5 + t^4 + t^3 +
t^2 + t + 1, 2, -1/2*t^5 - 1/4*t^3 - 1/4> ] > [X(A) eq &+[K| x : x in
PC(A)] : A in AA]; // gives "true" 42 times, checking the routines don't
crash.
```

The above analyses existing examples, and can be quickly programmed to analyse the Hilbert series of the 7555 hypersurfaces. I want to use these ideas to predict new examples. The nicely generated CYs are sparse in the set of all possible Hilbert series, so that if I modify $P_Y(t)$ by a clumsy amount, it is unlikely that the new $P_Y$ will correspond to a $Y$ that we can work with. However, I can modify the above `Num_r` to make more delicate variations, e.g.,

```
> A := [1,4,5,5,5,20];
```

is a hypersurface with curve $\Gamma$ of $1/5(1,4)$ contributing

$$(2t^9 + 4t^8 + 6t^7 + 2t^6 + 6t^5 + 4t^4 + 2t^3)/[1,1,5,5]$$

I modify it by changing the $6,2,6$ in the middle to $5,4,5$ (which should remove a generator in deg 5, and leave the point singularities and the degree of the curve unchanged).

```
> P1 := (1-t^20)/Denom([1,4,5,5,5]); // the Hilbert series of Y(20) in PP(1,4,5
> P2 := P1 + (-t^7+2*t^6-t^5)/Denom([1,1,5,5]);
> P2*Denom([1,4,5,5,6,9]); t^30 - t^18 - t^12 + 1
```

That is, the modification gives $Y(12,18) \subset \mathbb{P}(1,4,5,5,6,9)$.

Or change the $4,6,2,6,4$ in the middle to $3,6,4,6,3$ (which should remove a generator in deg 4 and add one in deg 6).

```
 > P3
:= P1 + (-t^8+2*t^6-t^4)/Denom([1,1,5,5]);
```

gives a plausible codim 4 guy with $9 \times 16$ resolution.

Or change the Numerator of the $\Gamma$ term to

$$(2t^9 + 3t^8 + 7t^7 + 2t^6 + 7t^5 + 3t^4 + 2t^3)$$

```
 > P4 := P1 -
(t^8+-t^7-t^5+t^4)/Denom([1,1,5,5]); > P4*Denom([1,5,5,5,7,8,9]); -t^40
+ t^26 + t^25 + t^24 + t^23 + t^22 - t^18 - t^17 - t^16 - t^15 - t^14 +
1
```

gives codim 3 candidate $Y \subset \mathbb{P}(1,5,5,5,7,8,9)$ with $5 \times 5$ Pfaffian matrix of degrees

```
 5,6,7,8
7,8,9 9,10 11
```

We can play this game about a million times with the existing list.

```
> load "KS2"; // Loading "KS2" > #AAKS2; // 14817
```

```
function BB(AA,r,a) // find transverse 1/r(a,r-a) curves in list AA of
CYs BB0 := [A : A in AA | (([r,a,r-a] in C) or ([r,r-a,a] in C)) where
B,C is Bask(A)]; BB1 := [BB0[i] : i in [1..#BB0] | &and[GCD(b[1],r) eq 1
: b in Bask(BB0[i])]]; return [BB1[i] : i in [1..#BB1] | (BB1[i][6] mod
r) in [0,a,r-a]]; end function;
```

```
> AA := BB(AAKS2,5,2); > #AA; // there are 19 of them
```

# 6 More ice cream

The second term in [BSz], Cor 3.3 is the sum

```
function Second(r,k) return &+[ ikbar*(r-ikbar)*(r-2*ikbar)*t^i where
ikbar is (i*k mod r) : i in [1..r-1]] / (6*r*(1-t^r)); end function;
```

I subtract a little initial term from it to get rid of term in $t$ and $t^2$, of the form

$$\frac{at + bt^2 + at^3}{(1-t)^4}.$$

```
 Second(r,k)-(r-2*k)*(r-k)*k*(t+t^3)/6/r/(1-t)^4
+((r-2*k)*2*ka*(r+2*ka)*t^2/6/r/(1-t)^4 where ka is Min(k,r-k));
```

This gives nice quantities like

$$5,2 \mapsto \frac{3t^5 - 2t^4 + 3t^3}{[1,1,1,5]} \quad \text{and}$$
$$8,3 \mapsto \frac{(8t^7 - 12t^6 + 17t^5 - 12t^4 + 8t^3)(1+t)}{[1,1,1,8]}$$

Abstracting from that gives function called Nterm, of the form

$$\frac{\text{integral symm polynom supported in [3..r]}}{[1,1,1,r]}$$

with numerator Num uniquely determined by the condition

$$\text{Num} * (1-t^b)^2(1-t^r-b)/(1-t)^3 \equiv 1 + t^b \mod (1-t^r)/(1-t)$$

(note the side-step $k \mapsto b =$ inverse of $k$ mod $r$). It would be jolly convenient to be able to calculate directly in the ring $\mathbb{Q}[t]/((1-t^r)/(1-t))$. That is,

```
 function
Nterm(r,b) return t^2* (t^(r-2)*(1+t^b)*InverseMod(Denom([b,b,r-b]) div
(1-t)^3, ((1-t^r) div (1-t))) mod ((1-t^r) div (1-t)))/Denom([1,1,1,r]);
end function;
```

The $t^2 * t^{r-2}$ is just a device for shifting the support into the interval $[3,\dots,r]$. I can test this as much as I like:

```
 for i in [1..20] do r := Random(500); k := Random(r);
if GCD(r,k) eq 1 then b := InverseMod(k,r);
r,k,Second(r,k)-(r-2*k)*(r-k)*k*(t+t^3)/6/r/(1-t)^4
+((r-2*k)*2*ka*(r+2*ka)*t^2/6/r/(1-t)^4 where ka is Min(k,r-k)) eq
Nterm(r,b); end if; end for;
```

(That looks trivially easy, but it took me 3 days of arm-wrestling with the computer to get it to work.) I think the $\deg\Gamma$ terms are basically simpler; I hope this concludes the treatment of pure $1/r(a,r-a)$ curves.

# 7 More notes on ice-cream

The two methods we have are working with are guessing from families of examples, and figuring out how to sum the expressions in [B], [BSz] in closed form. For the former consider $\mathbb{P}(r,\dots,r,a_1,a_2,\dots,a_n)$ with $r$ repeated $d+1$ times and the $a_i$ coprime to $r$ and to each other. This has a pure locus $\mathbb{P}^d$ of transvers type $1/r(a_1,a_2,\dots,a_n)$ and only isolated points Then you can subtract off all the junk, just leaving you with the contribution of the pure $1/r$ locus (of any dimension, any transverse cyclic orbifold). This lets us step back a bit from the $N_C$ term of [B], [BSz], which is possibly confusing because it treats the two isotypical normal bundles as sum and difference, rather than two separate terms. Experiments suggest that the contributions should be some kind of compound ice cream function (maybe double cone), with the main ingredients, as with $P_{\mathrm{orb}}$, derived from things like `InverseMod(ph(i), ph(r))` .

For example $\mathbb{P}(r,r,j)$ with $r$ coprime to 3 and $j \equiv 3 \bmod r$. Set $X$ to be the actual Hilb series minus the $P_{\mathrm{orb}}$ term for the isolated point $1/j(r,r)$. I treat $X$ as the unknown to be investigated. Subtract off the term `Qorb(r,[j],kP+r)/(1-t^r)` that corresponds to cutting by a zero-dim section of the $\mathbb{P}^1$, and is the only part of the formula with denominator $\prod_{a\in[1,r,r]}(1-t^a)$.

```
X := 1/Denom(A) - Qorb(j,[r,r],kP) - Qorb(r,[j],kP+r)/(1-t^r);
```

**Example 27** The following function calculates the contribution of a pure curve $\mathbb{P}(r,r)$ of transverse type $1/r(3)$ for $r = 1$ or $5 \bmod 6$. (The given routine is for $\mathbb{P}(25,25,103)$.)

```
 // i = 3 mod r, works for r = 1 or 5 mod 6
r:=25; a:=2; i:=3; j:=2*r*a+i; A:=[r,r,j]; kP := -&+A;
X:=1/Denom(A)-Qorb(j,[r,r],kP);
X-Qorb(r,[j],kP+r)/(1-t^r)
 + ( -(-1)^((r mod 6) div 3) *
  InverseMod(t^((r+1) div 2)*ph(3),ph(r))
  + a*((1+t^3)*InverseMod(t*ph(3)^2,ph(r)) mod ph(r)) )
 /t^(r*(a+1)-1)/Denom([1,1,r]);
```

Practically the same routine works for $r = 2$ or $4 \bmod 6$, with the term `InverseMod(t^((r+1) div 2)*ph(3),ph(r))` replaced by

```
((1+t)*InverseMod(t^((r+2) div 2)*ph(3),ph(r)) mod ph(r)).
```

There should be a more systematic solution without the case division.

## 7.1 Preliminary draft:

You get functions with numerators that for `PP(11,11,376)` look like:

$$154t^9 + 17t^8 + 120t^7 + 51t^6 + 86t^5 + 86t^4 + 51t^3 + 120t^2 + 17t + 154$$

This a sum of two arithmetic progressions with step $t^2$ starting out from the ends. Dividing into classes mod $2r$ gives them as sums of two more-or-less sensible terms: write $r = r_1 + r_2$ with $r_i = \frac{r\pm1}{2}$. Then the first term is $(1 \pm t^{r_1})(1 \mp t^{r_2})/(1-t^2)$. The second is something that sums in closed form like

$$\frac{-9t^{12} - 10t^{11} + t^{10} - t^2 + 10t + 9}{(1-t^2)(1+t)}.$$

I hope we'll eventually get this in into a more convincing form.

```
r:=11; a:=17; i:=2; j:=2*r*a+i;
A:=[r,r,j]; kP := -&+A; X:=1/Denom(A)-Qorb(j,[r,r],kP);
X - Qorb(r,[j],kP+r)/(1-t^r)
 + ((&*[1-(-t)^a : a in L] div (1-t^2) where L:=[(r+1) div 2,(r-1) div 2])
 + a*(((r-2)*(1-t^(r+1))+(r-1)*(t-t^r)-(t^2-t^(r-1))) div
 +((1-t^2)*(1+t))))/t^((a+1)*r-1)/Denom([1,1,r]);
```

That did $\mathbb{P}(11, 11, 375)$, but it works for all odd $r$ and all $a$.

# References


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