Plurigenera and invariants of canonical n-folds

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Abstract

My initial aim is to give formulas for the plurigenera of canonical n-folds for all n depending on integral parameters without implicit congruences. Understanding the coefficients in these formulas leads to a form of Hirzebruch RR for $\chi(D)$ that takes account of the $(-1)^n$ symmetry under $D \mapsto K - D$ arising from Serre duality and is primitive integral.

1 Introduction

As is well-known, the binomial coefficients $\binom{a}{n}$ for $n = 0, 1, 2, \dots$ form a $\mathbb{Z}$-basis for integral polynomial functions $\mathbb{Z} \to \mathbb{Z}$; they start with $\binom{a}{0} = 1$ and $\binom{a}{1} = a$

Among these, the binomial coefficients $\binom{a+m-1}{2m}$ are symmetric under $a \mapsto 1-a$, and form a $\mathbb{Z}$-basis for such functions; the first cases are $\binom{a-1}{0} = 1$ and $\binom{a}{2} = \frac{1}{2}a(a-1)$. In the same way, the functions

$$\binom{a+m}{2m+1} + \binom{a+m-1}{2m+1} = \frac{2a-1}{2m+1}\binom{a+m-1}{2m}, \tag{1}$$

for $m = 0, 1, 2, \dots$ base the integral polynomial functions that are skew under $a \mapsto 1-a$, starting with $\binom{a}{1} + \binom{a-1}{1} = 2a-1$ and $\binom{a+1}{3} + \binom{a}{3} = \frac{1}{6}(2a-1)a(a-1)$.

Binomial coefficients are made to be summed as generating functions: the Hilbert series of the polynomial ring $k[x_0, \dots, x_n]$ or of $\mathbb{P}^n$ with its usual polarisation $\mathcal{O}_{\mathbb{P}^n}(1)$ is

$$\sum_{a \geq 0} h^0(\mathbb{P}^n, \mathcal{O}(a))t^a = \sum_{a \geq 0} \binom{n+a}{n} t^a = \frac{1}{(1-t)^{n+1}}\,; \tag{2}$$

one can view this formula as the definition of the binomial coefficients. The same trick applied to my symmetric and skew guys gives:

Proposition 1.1 *Define* $A_n(t) =$

$$\begin{cases} \sum_{a\ge0} \binom{a+m-1}{2m} t^a = \dfrac{t^{m+1}}{(1-t)^{2m+1}} & \text{for } n = 2m; \\ \sum_{a\ge0} \left(\binom{a+m}{2m+1} + \binom{a+m-1}{2m+1}\right) t^a = \dfrac{(1+t)t^m}{(1-t)^{2m+2}} & \text{for } n = 2m+1. \end{cases} \tag{3}$$

I write out the starting values:

$$A_{-2} = 1 - t, \quad A_{-1} = 1 + t, \quad A_0 = \frac{t}{1-t}, \quad A_1 = \frac{t+t^2}{(1-t)^2}\,. \tag{4}$$

Each term $A_n(t)$ *obeys the functional equation*

$$tA_n(\tfrac{1}{t}) = (-1)^{n+1} A_n(t), \tag{5}$$

that I call Gorenstein symmetry, *or more precisely, Gorenstein symmetry of* parity $(-1)^{n+1}$ *and* canonical weight 1 *(the exponent of* t *in* (5)*).*

Proof The first sum for even m is evaluated using (2) with $n = 2m$; the shift from $n+a = 2m+a$ down to $a+m-1$ leads to the t^{m+1} in the numerator. On substituting $t \mapsto 1/t$, the numerator gets divided by t^{2m+2}, whereas the denominator gets divided by t^{2m+1} and multiplied by $(-1)^{2m+1} = -1$. The second sum is similar. □

Trivial as this is, these function are the key to RR and to the plurigenus formula for canonical n-folds. By the Hilbert Syzygies Theorem, a coherent sheaf on $\mathbb{P}^n$ is K-equivalent to a virtual sum of line bundles $\mathcal{O}_{\mathbb{P}^n}(a)$, so its Hilbert series is a sum of terms (2) times powers of t. The result for plurigenera of canonical n-folds is as follows:

Theorem 1.2 *Let* X *be a nonsingular* n*-fold over* $\mathbb{C}$ *with ample canonical class* K_X*. Write*

$$P_a(X) = h^0(aK_X) \quad \text{and} \quad P_{X,K_X}(t) = \sum_{a=0}^{\infty} P_a t^a \tag{6}$$

for its plurigenera and Hilbert series. Then

$$P_{X,K_X}(t) = \sum_{\substack{j\in[-2,n] \\ j\equiv n \bmod 2}} \beta_j A_j + qt \tag{7}$$

where the A_j are as in (3), the β_j are certain integers, and the correction term in degree 1 involves $q = 1 + p_g - (-1)^n\chi$.

Equivalently, the Hilbert polynomial for the plurigenera P_a for $a \geq 2$ is

$$P_a(X) = \begin{cases} \sum_{j=0}^{m} \beta_{2j} \binom{a+j-1}{2j} & n = 2m; \\ \sum_{j=0}^{m} \beta_{2j+1} \left(\binom{a+j}{2j+1} + \binom{a+j-1}{2j+1} \right) & n = 2m+1. \end{cases} \tag{8}$$

It depends on the $[\frac{n}{2}]+1$ coefficients $\beta_0, \beta_1, \dots, \beta_{[\frac{n}{2}]}$ or on the initial $[\frac{n}{2}]+1$ plurigenera.

Note The integer invariants β_i are basically versions of the Todd genera, but symmetrised under $D \mapsto K_X - D$; they play a starring role in the discussion below.

I show below how to express the coefficients β_j in terms of Chern numbers. However, it is clear a priori that they are integers, and are determined by the first $\left[\frac{n}{2}\right]+1$ plurigenera. This follows because the leading term in t of the successive terms of (8) are $1, t, t^2, \dots,$ increasing by one at each step, so that the β_j term contributes to the plurigenus $P_{[\frac{j}{2}]+1}$. See the examples below.

1.1 Initial terms of the power series

Split into even and odd cases, (7) gives $P_{X,K_X}(t) =$

$$1 - t + qt + \chi(\mathcal{O}_X)\frac{t}{1-t} + \beta_2\frac{t^2}{(1-t)^3} + \cdots + K^n\frac{t^{m+1}}{(1-t)^{n+1}}, \tag{9}$$

$$1 + t + qt - \chi(\mathcal{O}_X)\frac{t+t^2}{(1-t)^2} + \beta_3 A_3 + \cdots + \frac{K^n}{2}A_{2m+1}. \tag{10}$$

It is worth spelling out how the statement deals with the initial terms, ensuring that the series starts with $1+p_g t$. In the even case, I set $\beta_{-2} = 1$ and $\beta_0 = \chi(\mathcal{O}_X) = \mathrm{Td}_X$. In the odd case, $\beta_{-1} = 1$ and $\beta_1 = -\chi(\mathcal{O}_X) = -\mathrm{Td}_X$. It is clear from (4) that the negative terms $A_{-2} = 1 - t$ and $A_{-1} = 1 + t$ are linear polynomials, so the coefficients β_{-2} or β_{-1} only affect the $t^0 = 1$ and t terms in $P_X(t)$, and do not contribute to the Hilbert polynomial. The irregularity term qt in (7) with $q = 1 + p_g - (-1)^n\chi$ serves only to adjust the term in t from its RR value $(-1)^n\chi - 1$ to $P_1 = p_g$. Kodaira vanishing gives $P_a(X) = h^0(aK_X) = \chi(aK_X)$ for $a \geq 2$, so no further irregularity adjustment is needed.

1.2 Higher terms and growth of the power series

At the top end, the growth of plurigenera is controlled by the leading term

$$\begin{cases} K^n\binom{a+m-1}{2m} & \text{for } n = 2m; \\ \frac{K^n}{2}\left(\binom{a+m}{2m+1} + \binom{a+m-1}{2m+1}\right) & \text{for } n = 2m+1, \end{cases} \tag{11}$$

so that $\beta_n = K^n$ in the even case and $\beta_n = \frac{K^n}{2}$ in the odd case. The formula implies that K^n is divisible by 2 for n odd.

The next-to-leading term is

$$\beta_{n-2}\binom{a+m-2}{2m-2} \quad \text{with} \quad \beta_{n-2} = K^{n-2}\cdot\frac{1}{12}(mK^2 + c_2) \tag{12}$$

for $n = 2m$ or

$$\beta_{n-2}\left(\binom{a+m-2}{2m-1} + \binom{a+m-3}{2m-1}\right) \tag{13}$$

$$\text{with} \quad \beta_{n-2} = K^{n-2}\cdot\frac{1}{24}((m-1)K^2 + c_2)$$

for $n = 2m + 1$. The second factor in each product has the same terms as Td_2. With hindsight, one might try to view it as Td_2 plus binomial coefficient times K^2 times Td_0.

1.3 Symmetric RR

Writing the coefficients β_j as combinations of Chern numbers of X leads directly to a symmetric form of the Hirzebruch RR formula, where D appears only in *pseudo-binomial coefficients* defined by

$$\binom{D+(m-1)K, K}{2m} = \frac{(D+(m-1)K)(D+(m-2)K)\cdots(D-mK)}{(2m)!} \tag{14}$$

for $n = 2m$, with $2m$ factors that are reflected and multiplied by -1 by the substitution $D \mapsto K - D$, and

$$\frac{1}{2}\left(\binom{D+mK, K}{2m+1} + \binom{D+(m-1)K, K}{2m+1}\right)$$
$$= (D - \tfrac{1}{2}K) \times \frac{(D+(m-1)K)(D+(m-2)K)\cdots(D-mK)}{(2m+1)!} \qquad (15)$$

for $n = 2m+1$, with $2m+1$ terms that are again reflected and multiplied by -1 by $D \mapsto K - D$.

The result will be: in the even case $n = 2m$

$$\operatorname{ch}(D) \cdot \operatorname{Td}_X = \sum_{j=0}^{m} B_{2j} \cdot b_{n-2j} \qquad (16)$$

where b_0, b_2, b_4, etc., are like Td_0, Td_2, Td_4 but modified with binomial coefficients times K^2 times lower Tds.

In the odd case $n = 2m+1$ we get, for example,

$$n = 3: \quad \operatorname{ch}(D) \cdot \operatorname{Td}_X = \left(D - \frac{K}{2}\right)\left(\frac{D(D-K)}{6} + \frac{c_2}{12}\right) \qquad (17)$$

and

$$n = 5: \quad \operatorname{ch}(D) \cdot \operatorname{Td}_X =$$
$$\left(D - \tfrac{K}{2}\right)\left(\tfrac{(D+K)D(D-K)(D-2K)}{5!} + \tfrac{D(D-K))}{3!} \cdot \tfrac{c_1^2+c_2}{12} + \operatorname{Td}_4\right) \qquad (18)$$

2 Index 1

The plurigenera and Hilbert series of a canonical n-fold X are given by

$$P_i(X) = H^0(iK_X) \quad \text{and} \quad P_{X,K_X}(t) = \sum_{i=0}^{\infty} P_i t^i. \qquad (19)$$

Assume first for simplicity that X has local index 1 (that is, only Gorenstein canonical singularities) and is regular, so that $\chi(\mathcal{O}_X) = 1 + (-1)^n p_g$. (More general cases later.)

It is known that the Hilbert series is a rational function with denominator $(1-t)^{n+1}$, and Serre duality implies it is Gorenstein symmetric of degree 1, meaning that

$$tP(1/t) = (-1)^{n+1}P(t). \tag{20}$$

Thus

$$P(t) = \frac{\text{Num}}{(1-t)^{n+1}} = \frac{1 + a_1t + a_2t^2 + \cdots + a_2t^n + a_1t^{n+1} + t^{n+2}}{(1-t)^{n+1}}, \tag{21}$$

with Num a symmetric polynomial of degree $n+2$; that is, Num is palindromic, with $a_p = a_q$ for $p+q = n+2$. It follows that $P(X, K_X)$ is determined as a linear combination of just $\left[\frac{n}{2}\right]+1$ coefficients. Whereas the RR formula $\chi(iK) = \text{ch}(iK)\cdot\text{Td}_X$ involves $n+1$ terms. There are several different strategies to choose these invariants, with different advantages:

(i) The Hilbert polynomial $\text{Hp}(i)$, with $\text{Hp}(i) = h^0(iK)$ for all $i \geq 2$.

(ii) The coefficients of $\text{Hp}(i)$ are the $n+1$ terms $K^{n-q}\cdot\text{Td}_{X,q}$ for $q = 0, \dots, n$, corresponding to all the Todd classes

$$\big[\text{ch}(iK)\cdot\text{Td}(X)\big]_n = \sum_{q=0}^{n} \frac{(iK)^q}{q!}\cdot\text{Td}_{n-q}\,. \tag{22}$$

(iii) The palindromic coefficients $1, a_1, a_2, \dots, a_2, a_1, 1$ in the numerator of (21).

(iv) The initial plurigenera P_k for $k = 1, 2, \dots, \left[\frac{n}{2}\right]+1$.

(v) The "principal parts" of $P(t)$ viewed as a meromorphic function with pole at $t=1$. In more detail, P has the expansion $P(t) =$

$$\begin{cases} \displaystyle\sum_{j=-1}^{m} b_{2j}\frac{t^j}{(1-t)^{2j+1}} = (1-t) + \cdots + K^n\frac{t^{m+1}}{(1-t)^{n+1}} & \text{if } n = 2m, \\ \displaystyle\sum_{j=0}^{m} \frac{b_{2j}}{2}\frac{t^j + t^{j+1}}{(1-t)^{2j}} = (1+t) + \cdots + \frac{K^n}{2}\frac{t^m + t^{m+1}}{(1-t)^{n+1}} & \text{if } n = 2m-1 \end{cases} \tag{23}$$

having $\left[\frac{n}{2}\right]+1$ coefficients b_{2j} for $j = -1, \dots, m$ or $j = 0, \dots, m$.

(vi) Golyshev [Go] views the *roots* of the Hilbert polynomial $\mathrm{Hp}(i)$ as basic invariants. These are real algebraic numbers in a "canonical strip" centred at $n = \frac{1}{2}$, and complex conjugate pairs on the line $\mathrm{Re}\, z = 1/2$). This view may provide information akin to the Bogomolov–Miyaoka–Yau inequality and its generalisations.

The three sets of invariants (iii–v) relate to one another in obvious ways, and each gives integral invariants without hidden congruence properties. (iii) and (v) display the symmetry and dependence on $\left[\frac{n}{2}\right]+1$ parameters very neatly.

The nice feature of (v) is that each term has the same Gorenstein symmetry (20). It also translates directly into a neat formula for the plurigenera, as follows. You recognise $\frac{1}{(1-t)^{n+1}}$ as the Hilbert series of $\mathbb{P}^n, \mathcal{O}(1)$, with its well-known Hilbert polynomial $\binom{n+i}{n}$, so that one deduces practically without calculation that in the even dimensional case

$$\begin{aligned} P_i(X) &= \sum_{j=0}^{m} \beta_{2m-2j} K^{2j} \binom{i+j-1}{2j} \\ &= \chi + K^2 \beta_{2m-2} \binom{i}{2} + \cdots + K^n \binom{i+m-1}{n} \quad \text{for } i \geq 2 \end{aligned} \tag{24}$$

where $n = 2m$, and in the odd dimensional case

$$\begin{aligned} &\sum_{j=1}^{m} \beta_{2m-2j} \frac{K^{2j-1}}{2} \left(\binom{i+j-1}{2j+1} + \binom{i+j}{2j+1} \right) \\ &= (-\chi) \cdot (2i-1) + \cdots + \frac{K^n}{2} \left(\binom{i+m}{n} + \binom{i+m-1}{n} \right) \quad \text{for } i \geq 2 \end{aligned} \tag{25}$$

if $n = 2m - 1$

Serre duality and symmetric RR

Serre duality $\chi(\mathcal{O}_X(K_X - D)) = (-1)^n \chi(\mathcal{O}_X(D))$ gives that RR is $(-1)^n$ symmetry under $D \mapsto K_X - D$, so that $\mathrm{Hp}(i)$ is $(-1)^n$ symmetric under $i \leftrightarrow 1 - i$. This symmetry implies (for example) that Hp is a function of $\binom{i}{2}$ for even n, and equals $2i - 1$ times a function of $\binom{i}{2}$ for odd n; hence it depends on $\left[\frac{n}{2}\right] + 1$ coefficients.

The formula

$$\mathrm{ch}(D)\cdot \mathrm{Td}_X = e^D \times \prod \frac{x_i}{1-e^{-x_i}} \tag{26}$$

in terms of products of the Chern roots x_i is identically equal to

$$e^{D-\frac{K}{2}} \times \prod \frac{x_i}{2\sinh\frac{x_i}{2}} = e^{D-\frac{K}{2}} \times \mathrm{Td}^+_X, \tag{27}$$

(take $e^{\frac{K}{2}} = e^{-\sum\frac{x_i}{2}}$ inside the product, then use $\frac{xe^{-\frac{x}{2}}}{1-e^{-x}} = \frac{x}{e^{\frac{x}{2}}-e^{-\frac{x}{2}}} = \frac{x}{2\sinh\frac{x}{2}}$), where the seed power series is the even function

$$\frac{s}{2\sinh\frac{s}{2}} = 1 - \frac{1}{24}s^2 + \frac{7}{5760}s^4 - \frac{31}{967680}s^6 + \frac{127}{154828800}s^8 - \cdots. \tag{28}$$

The modified Todd classes only appear in even degrees, making the symmetry under $D \mapsto K_X - D$ obvious, at the cost of extra powers of 2 in the denominator.

However, the individual RR terms $K^j \cdot \mathrm{Td}_{n-j}$ in $\mathrm{ch}\cdot\mathrm{Td}$ do not know about the symmetry: rendering the right-hand side in terms of the elementary symmetric functions $c_i = \sigma_i(x_1,\dots,x_n)$ completely jumbles up the symmetry, and the symmetry of RR involves different coincidences in each dimension among the terms in the Todd classes.

In addition, the Chern numbers have hidden divisibility properties, such as the well-known fact that $K^2 + c_2$ is divisible by 12 in dimension 2, or K^3 divisible by 2 and Kc_2 divisible by 24 in dimension 3 (and later, divisibility conditions by 720 or 30480; see **??**, below). I want to write the plurigenus directly with integer coefficients and without hidden congruences.

3 Pseudo-binomial coefficients and symmetric RR

There is an better solution. I define the pseudo-binomial coefficients: for even $n = 2m$, set

$$\binom{D+(m-1)K, K}{2m} = \frac{(D+(m-1)K)(D+(m-2)K)\cdots(D-mK)}{(2m)!} \tag{29}$$

so that $D \mapsto K - D$ inverts the $2m$ terms and multiplies each by -1. For odd $n = 2m+1$, set

$$\begin{pmatrix} D+mK, K \\ 2m+1 \end{pmatrix} = \frac{1}{2}\left(\begin{pmatrix} D+mK, K \\ 2m+1 \end{pmatrix} + \begin{pmatrix} D+(m-1)K, K \\ 2m+1 \end{pmatrix}\right)$$

$$= (D - \tfrac{1}{2}K) \times \frac{(D+(m-1)K)(D+(m-2)K)\cdots(D-mK)}{(2m+1)!} \quad (30)$$

If we write RR in terms of symmetrised Todd classes Td^+ and these pseudo-binomial coefficients then each term is symmetric under $D \mapsto K - D$.

3.1 Symmetric RR

Writing the coefficients β_j as combinations of Chern numbers of X leads directly to a symmetric form of the Hirzebruch RR formula, where D appears only in *pseudo-binomial coefficients* defined by

If $n = 2m$:

$$B_n(D) = \begin{pmatrix} D+(m-1)K, K \\ 2m+1 \end{pmatrix} \quad (31)$$

$$= \frac{(D+(m-1)K)(D+(m-2)K)\cdots(D-mK)}{(2m)!} \quad (32)$$

If $n = 2m+1$:

$$B_n(D) = \frac{1}{2}\left(\begin{pmatrix} D+mK, K \\ 2m+1 \end{pmatrix} + \begin{pmatrix} D+(m-1)K, K \\ 2m+1 \end{pmatrix}\right) \quad (33)$$

$$= (D - \tfrac{1}{2}K) \times \frac{(D+(m-1)K)(D+(m-2)K)\cdots(D-mK)}{(2m+1)!} \quad (34)$$

In either case $D \mapsto K - D$ reflects the terms and multiplies each by -1.

The result corresponding to Theorem 1.2 is:

Theorem 3.1 *In the even case $n = 2m$*

$$\mathrm{ch}(D) \cdot \mathrm{Td}_X = \sum_{j=0}^{m} B_{2j} \cdot b_{n-2j} \quad (35)$$

where b_0, b_2, b_4, etc., are like Td_0, Td_2, Td_4 *but modified with binomial coefficients times* K^2 *times lower* Tds.

In the odd case $n = 2m+1$ we get, for example,

$$n = 1: \quad \mathrm{ch}(D) \cdot \mathrm{Td}_X = B_1(D) = D - \frac{K}{2} \quad (= 1 - g + \deg D) \tag{36}$$

$$n = 3: \quad \mathrm{ch}(D) \cdot \mathrm{Td}_X = B_3(D) + B_1(D) \cdot \frac{c_2}{12} \tag{37}$$

$$= \left(D - \frac{K}{2}\right)\left(\frac{D(D-K)}{3!} + \frac{c_2}{12}\right) \tag{38}$$

$$n = 5: \quad \mathrm{ch}(D) \cdot \mathrm{Td}_X = B_5(D) + B_3(D) \cdot \frac{1}{12}(c_1^2 + c2) + B_1(D) \cdot b_4 \tag{39}$$

where $b_4 = \frac{1}{720}(-c_1^2 c_2 + c_1 c_3 + 3c_2^2 - c_4)$. *Note that this is not exactly equal to* $\mathrm{Td}_4 = \frac{1}{720}(-c_1^4 + 4c_1^2 + c_1 c_3 + 3c_2^2 - c_4)$.

```
X := &+[D^(5-i)*Todd[i+1]/Factorial(5-i) : i in [0..5]];
X div (D-K/2) - (D+K)*D*(D-K)*(D-2*K)/Factorial(5)
  - D*(D-K)/Factorial(3)*1/12*(c1^2+c2)
  - (-1/720*c1^2*c2 + 1/720*c1*c3 + 1/240*c2^2 - 1/720*c4);
```

1. Curve of genus g

$$P(X, K_X) = 1 + gt + \sum (2i-1)(g-1)t^i = (-\chi) \cdot \frac{t + t^2}{(1-t)^2} + (1+t), \tag{40}$$

where $-\chi = g - 1 = -\mathrm{Td}_1 = \frac{1}{2}K_X$. Obviously, this gives

$$P_i = (2i-1)(g-1) \quad \text{for } i \geq 2. \tag{41}$$

where $-\chi = g - 1 = -\mathrm{Td}_1 = \frac{1}{2}K_X$.

2. Surface

RR and Kodaira vanishing gives

$$P_m = \begin{cases} 1 & \text{if } n = 0, \\ p_g & \text{if } n = 1, \\ \chi + K^2 \cdot \binom{n}{2} & \text{if } n \geq 2. \end{cases} \tag{42}$$

This gives the well-known Hilbert series

$$P(X, K_X) = \frac{1 + (p_g - 3)t + (K^2 - 2p_g + 4)t^2 + (p_g - 3)t^3 + t^4}{(1-t)^3} + qt \quad (43)$$

$$= 1 - t + \chi \cdot \frac{t}{1-t} + K^2 \cdot \frac{t^2}{(1-t)^3} \,. \quad (44)$$

3-fold

The Hilbert series is

$$P(X, K_X) = 1 + t + qt + (-\chi) \cdot \frac{t + t^2}{(1-t)^2} + \frac{K^3}{2} \cdot \frac{t^2 + t^3}{(1-t)^4} \quad (45)$$

giving

$$P_a = (-\chi) \cdot (2a - 1) + \frac{K^3}{12} \cdot a(a-1)(2a-1) \quad \text{for } a \geq 2. \quad (46)$$

Here $\chi = \frac{1}{24} c_1 c_2 = -\frac{1}{24} K c_2$. Note that c_3 does not appear in this world. In terms of $p_g = P_1 = 1 - \chi$ and $P_2 = \frac{K^3}{2} - 3\chi$, we get $\chi = 1 - p_g$ and $K^3 = 2P_2 - 6\chi$.

e.g. $K^3 = 6$, $\chi = -4$ (corresponding to $p_g = 5$) gives $X_6 \subset \mathbb{P}^4$.

$K^3 = 2$, $\chi = -3$ (corresponding to $p_g = 4$) gives $X_{10} \subset \mathbb{P}(1, 1, 1, 1, 5)$, the double cover of $\mathbb{P}^3$ branched in S_{10}.

$K^3 = 8$, $\chi = -4$ (corresponding to $p_g = 5$) gives $X(4, 4) \subset \mathbb{P}^5(1^5, 2)$.

```
// check in Magma
K3 := 6; chi := -4;
P := K3/2*(t^2+t^3)/(1-t)^4 - chi*(t+t^2)/(1-t)^2 + (1+t); P;
```

4-fold

$$P(X, K_X) = 1 - t + \chi \cdot \frac{t}{1-t} + \beta_2 \cdot \frac{t^2}{(1-t)^3} + K^4 \cdot \frac{t^3}{(1-t)^5} \,, \quad (47)$$

where $\beta_2 = K^2 \cdot b_2$ and $b_2 = \frac{1}{12}(2K^2 + c_2)$, giving

$$P_a = \chi + \beta_2 \binom{a}{2} + K^4 \binom{a+1}{4}. \quad (48)$$

Take the 3 parameters to be K^4, β_2 and $\chi = \mathrm{Td}_4 = p_g + 1$. This has the advantage that they are integers with no implicit congruences, directly related to the initial plurigenera. In terms of Chern numbers, you have to say that K^4, K^2c_2 and $\chi = \mathrm{Td}_4$ are integers with K^2c_2 even and $2K^4 + K^2c_2$ divisible by 12.

e.g. $K^4 = 7$, $p_g = 6$, $K^2c_2 = 154$ gives $X_7 \subset \mathbb{P}^5$.

$K^4 = 8$, $p_g = 6$, $K^2c_2 = 152$ gives $X(4,6) \subset \mathbb{P}(1^6, 3)$.

$K^4 = 6$, $p_g = 6$, $K^2c_2 = 156$ gives $X(3,8) \subset \mathbb{P}(1^6, 4)$.

```
// check in Magma
K4 := 7; chi := 7; dc2 := 154;
P := K4/6*(t^2 + 4*t^3 + t^4)/(1-t)^5 + dc2/12*t^2/(1-t)^3
      + chi*t/(1-t) + (1-t); P;

K4 := 6; chi := 7; dc2 := 156; // same line for P
P*(1-t)^6*(1-t^4); // t^11 - t^8 - t^3 + 1
```

5-fold

$$P(X, K_X) = 1 + t + (-\chi) \cdot \frac{t + t^2}{(1-t)^2} + \frac{K^3}{2} \cdot b_2 \cdot \frac{t^2 + t^3}{(1-t)^4} + \frac{K^5}{2} \cdot \frac{t^3 + t^4}{(1-t)^6} \quad (49)$$

Here $K^3 \cdot b_2 = \frac{1}{12}(K^5 + K^3c_2)$ (it is just a coincidence that $b_2 = \mathrm{Td}_2$). For $i \geq 2$ the coefficient of t^i is

$$\begin{aligned} P_i(X) = 1 &+ (-\chi) \cdot (2i - 1) \\ &+ \frac{K^3}{2} \cdot \mathrm{Td}_2 \cdot \left(\binom{i+1}{3} + \binom{i}{3} \right) + \frac{K^5}{2} \left(\binom{i+2}{5} + \binom{i+1}{5} \right). \end{aligned} \quad (50)$$

```
// check in Magma
K5 := 8; chi := -6; dTd2 := 20;
P := K5/2 * (t^3+t^4)/(1-t)^6
  + dTd2/2 * (t^2+t^3)/(1-t)^4
  - chi * (t+t^2)/(1-t)^2 + (1+t); P;
```

6-fold

$$P(X, K_X) = K^6 \cdot \frac{t^4}{(1-t)^7} + K^4\beta_2 \cdot \frac{t^3}{(1-t)^5} + K^2\beta_4 \cdot \frac{t^2}{(1-t)^3} + \chi \cdot \frac{t}{1-t} + 1 - t. \quad (51)$$

where

$$\beta_2 = \frac{1}{12}(3K^2 + c_2), \quad \beta_4 = \frac{1}{720}(6K^4 + 9K^2c_2 + c_1c_3 + 3c_2^2 - c_4) \quad (52)$$

Hence

$$P_i(X) = K^6 \cdot \binom{i+2}{6} + K^4\beta_2 \cdot \binom{i+1}{4} + K^2\beta_4 \cdot \binom{i}{2} + \chi \quad \text{for } i \geq 2. \quad (53)$$

8-fold

$$\chi(D) = \binom{D+3K, K}{8} + \binom{D+2K, K}{6}\beta_2 + \binom{D+K, K}{4}\beta_4 + \binom{D, K}{2}\beta_6 + \beta_8$$

where

$$\begin{aligned}
\beta_2 &= \frac{1}{12}(4c_1^2 + c_2), \\
\beta_4 &= \frac{1}{720}(18c_1^4 + 14c_1^2c_2 + c_1c_3 + 3c_2^2 - c_4), \\
\beta_6 &= \frac{1}{60280}(12c_1^6 + 30c_1^4c_2 + 12c_1^3c_3 + 32c_1^2c_2^2 - 12c_1^2c_4 + 11c_1c_2c_3 \\
&\quad - 2c_1c_5 + 10c_2^3 - 9c_2c_4 - c_3^2 + 2c_6) \\
\beta_8 &= \mathrm{Td}_8 = \frac{1}{3628800}(-3c_1^8 + 24c_1^6c_2 - 14c_1^5c_3 - 50c_1^4c_2^2 + 14c_1^4c_4 \\
&\quad + 26c_1^3c_2c_3 - 7c_1^3c_5 + 8c_1^2c_2^3 - 19c_1^2c_2c_4 + 3c_1^2c_3^2 + 7c_1^2c_6 \\
&\quad + 50c_1c_2^2c_3 - 16c_1c_2c_5 - 13c_1c_3c_4 + 3c_1c_7 + 21c_2^4 \\
&\quad - 34c_2^2c_4 - 8c_2c_3^2 + 13c_2c_6 + 3c_3c_5 + 5c_4^2 - 3c_8)
\end{aligned}$$

Each of those formulas in dimension 1–8, Serre duality involve different coincidences among the terms in the Todd classes.

4 Letter about `Qorb`

Shengtian corrected my Qorb formula, and you should just replace my version by hers:

```
http://www.warwick.ac.uk/~masda/Ice/My_magmarc
down 4 paragraphs, from line 35
```

My formula allowed the Proj to have 1-dimensional singular locus such as $\frac{1}{6}(1,2,3)$, but does not give the right result for singular locus of dimension ≥ 2 such as $\frac{1}{8}(1,2,2)$ or $\frac{1}{R}(1,r,r)$ with r divides R, or $\frac{1}{R}(1,ar,br)$ with $r = \mathrm{hcf}(abr, R)$. I think I take out the factor $(1-t^r)$ once only before doing XGCD, whereas she takes it out the right number of times.

What Qorb is doing is also becoming clearer to me. For a completely general graded ring (in positive degree, with $R_0 = k$), the Hilbert series $P(t)$ is a power series that is also a rational function. If you view it as a meromorphic function, it has poles at the primitive rth roots of unity, of order $\leq \dim \mathrm{Fix}\,\mu_r$, the dimension of the fixed locus of the subgroup of rth roots of unity $\mu_r \subset \mathbb{C}^\times$ acting on $\mathrm{Spec}\,R$. (This is practically obvious.)

Therefore the rational function has a partial fraction decomposition with denominators $(\Phi_r)^d$ (where Φ_r is the cyclotomic polynomial), corresponding to the principal parts of the meromorphic function. I don't know exactly at what level of generality, but in the cases we want, there is way of parsing the principal parts (at all the roots of 1, including 1 itself, the initial term P_I), as a sum of terms with denominator $\prod(1-t^r)$, as r runs through a set of orders of isotropy groups at each stratum and abutting strata, and numerator in a stated short support.

To see what I mean, just run Shengtian's Qorb on a few $\frac{1}{R}(a, br, cr)$ cases, for example

```
  P := Qorb(28,[5,7,14],2); P;
  Factorization(Denominator(P));
[
    <t - 1, 4>,
    <t + 1, 2>,
    <t^2 + 1, 1>,
    <t^6 - t^5 + t^4 - t^3 + t^2 - t + 1, 2>,
    <t^6 + t^5 + t^4 + t^3 + t^2 + t + 1, 3>,
```

```
    <t^12 - t^10 + t^8 - t^6 + t^4 - t^2 + 1, 1>
]
```

One sees that

$$\frac{(-t^36 + t^35 - t^34 + t^32 - t^31 + t^30 - t^29 + t^27 - t^26 + t^25 - t^23)}{\prod_{r\in[1,7,14,28]}(1-t^r)}$$

5 Coarse general statement

$R = k[x_0, \dots, x_n]/I$ is a graded ring in positive degrees. For M a finite graded R-module, write $P_M(t)$ for its Hilbert series. On the one hand, $P_M(t)$ is a power series with integral coefficients. On the other, it is a rational function with denominator a product of $(1 - t^{a_i})$ where $a_i = \operatorname{wt} x_i$.

Write $\operatorname{Fix}(\mu_r) \subset \operatorname{Spec} R$ for the fixed locus of $\mu_r \subset \mathbb{C}^\times$. Then $P_M(t)$ can also be viewed as a rational function with poles of order $\le d_r$ at the primitive rth roots of 1, where

$$d_r = \dim\bigl(\operatorname{Fix}(\mu_r) \cap \operatorname{Supp} M\bigr).$$

(NB: the dimension in affine space $\operatorname{Spec} R$, which is one more than the dimension in $\operatorname{Peoj} R$.) This is obvious by the standard hyperplane section argument. We can also view $P_M(t)$ as a meromorphic function with the same poles, and ask for his principal parts at each primitive rth root ε_r. Since the primitive rth roots of 1 are roots of the irreductible cyclotomic polynomial $\Phi_r(t)$, the principal part can be treated as

$$\frac{\text{Num}}{\Phi_r^{d_r}},$$

where Num is a polynomial with rational denominator. This gives a partial fraction decomposition of $P_M(t)$ as a sum of such terms.

Conjecture 5.1 *The same principal part is given by a unique best rational function*

$$\frac{\text{Num}}{\prod \Phi_{r_i}},$$

where Num *has integral coefficients, and the denominator involves the* r_i *with* $\operatorname{Fix}(\mu_{r_i})$ *adjacent to* $\operatorname{Fix}(\mu_r)$*. Here "best" can be interpreted as "shortest".*

Then $P_M(t)$ is a sum of such terms over finitely many values of r_i. This is a coarse result. It says nothing about relation with RR, but also it does not require any assumptions about smoothness etc.

6 Canonical 3-folds

It has been well understood for 30 years that in dimension ≥ 3, nonsingular is not enough. A simple example illustrates some interesting and typical features. Consider the general hypersurface $X_{10} \subset \mathbb{P}(1,1,2,2,3)$; write x_1, x_2, y_1, y_2, z for the coordinates. This has $5 \times \frac{1}{2}(1,1,1)$ orbifold points along $\mathbb{P}^1(y_1, y_2)$ and a $\frac{1}{3}(1,2,2)$ point at $P_z = (0,0,0,0,1)$.

I write his canonical Hilbert series as follows: first, the initial term

$$P_I = \frac{1 - 2t + 3t^2 + 3t^3 - 2t^4 + t^5}{(1-t)^4}, \tag{54}$$

takes care of $P_1 = 2$, $P_2 = 5$. Next, add in the orbifold terms

$$P_{\text{orb}}(\tfrac{1}{2}(1,1,1),1) = \frac{-t^3}{(1-t)^3(1-t^2)}, \quad P_{\text{orb}}(\tfrac{1}{3}(1,2,2),1) = \frac{-t^3 - t^4}{(1-t)^3(1-t^3)}. \tag{55}$$

to take care of the periodicity, getting

$$P_I + 5 \times P_{\text{orb}}(\tfrac{1}{2}(1,1,1),1) + P_{\text{orb}}(\tfrac{1}{3}(1,2,2),1) = \frac{1 - t^{10}}{\prod_{a \in [1,1,2,2,3]} (1 - t^a)} \tag{56}$$

References

[Go] V. Golyshev, The canonical strip, arXiv:0903.2076, 4 pp.