TROPICAL AFFINE MANIFOLDS

from MIRROR SYMMETRY to BERKOVICH GEOMETRY

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duality between
Calabi-Yau varieties
$(X, \check{X})$

non-archimedean
geometry

Geometric explanation for MS

Ex: $V(f_4) \subseteq \mathbb{P}^3$ K3 surface
$V(f_5) \subseteq \mathbb{P}^4$ quintic 3-fold

variety with a nowhere vanishing holomorphic n-form,
equiv: with trivial canonical line bundle

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Consider a projective family of complex Calabi-Yau varieties of dim n
$X \longrightarrow \Delta^* \subseteq \mathbb{C}_t$ such that X is maximally degenerate

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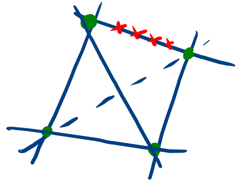
"meromorphic at 0": it extends to a projective family $\mathcal{X} \longrightarrow \Delta$
where $\mathcal{X}$ is smooth
and $\mathcal{X}_0$ is strict normal crossings

"as degenerate as poss": there is a non-empty intersection of n+1 comp's of $\mathcal{X}_0$

Ex: $\mathcal{X} = \{ t f_4 + x_0 x_1 x_2 x_3 = 0 \} \subseteq \mathbb{P}^3_x \times \mathbb{C}_t$

$\mathcal{X}_0$:



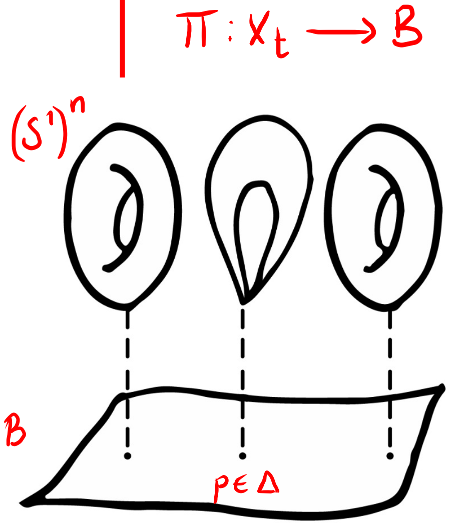
Geometric explanation for MS : SYZ conjecture

Consider a projective family of complex Calabi-Yau varieties of dim n

$X \longrightarrow \Delta^* \subseteq \mathbb{C}_t$ such that X is maximally degenerate

Then a general fibre X_t admits a fibration $X_t \longrightarrow B$

- to a topological manifold B,
- whose fibres are special Lagrangian real tori of dim n
- away from a locus Δ of codimension ≥ 2 in B



Idea: construct $\check{X}_t$ starting from dual torus fibration

$$\begin{tikzcd} X_t \arrow[dr, "\pi"'] & & X_t \arrow[dl, "\check{\pi}"] \\ & B & \end{tikzcd}$$

Base of the SYZ fibration

Given an SYZ fibration $\pi : X_t \to B$

- B is a topological manifold of $\dim_{\mathbb{R}} B = n$
- outside Δ, B admits an integral affine structure

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  $B \setminus \Delta$ admits an open cover $(U_i)_i$ and charts $(\varphi_i : U_i \longrightarrow \mathbb{R}^n)_i$ such that the transition functions $\varphi_i \circ \varphi_j^{-1} \in GL_n(\mathbb{Z})$

  (TROPICAL AFFINE MANIFOLDS)
- outside Δ, B admits a Monge-Ampère metric

Idea: describe B and its additional structure to construct mirror family

Topology of B : dual complex

$X \longrightarrow \Delta^* \subseteq \mathbb{C}_t$

$\mathcal{X} \longrightarrow \Delta \subseteq \mathbb{C}_t$

$\mathcal{X}_0$ snc

$D(\mathcal{X}_0)$ dual complex

$=$

Given $Y = \bigcup Y_i$ snc variety, the dual complex is a cell complex consisting of

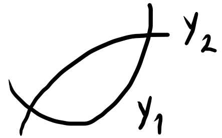
irr comp $Y_i \longleftrightarrow$ 0-cell v_i

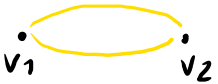
irr comp of $Y_i \cap Y_j \neq \emptyset \longleftrightarrow$ 1-cell $\langle v_i, v_j \rangle$

irr comp of $Y_{i_0} \cap \dots \cap Y_{i_k} \neq \emptyset \longleftrightarrow$ k-cell $\langle v_{i_0}, \dots, v_{i_k} \rangle$

Examples:

1)





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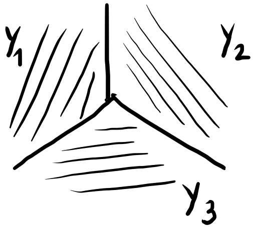
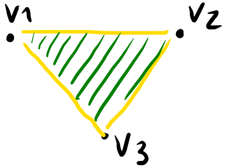
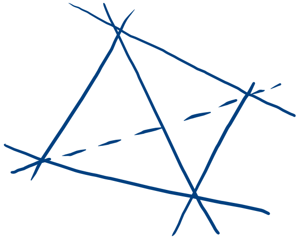
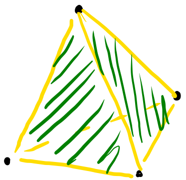




3) $\mathcal{X} = \{ t f_4 + x_0 x_1 x_2 x_3 = 0 \} \subseteq \mathbb{P}^3_x \times \mathbb{C}_t$

$\mathcal{X}_0 = (x_0 x_1 x_2 x_3 = 0) \subseteq \mathbb{P}^3_{\mathbb{C}}$





Topology of B : dual complex // Berkovich skeleton

$X \longrightarrow \Delta^* \subseteq \mathbb{C}_t$

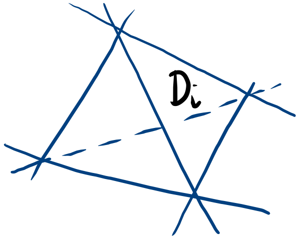
$\mathcal{X} \longrightarrow \Delta \subseteq \mathbb{C}_t$

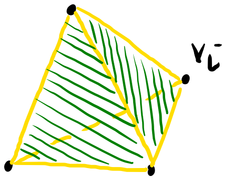
$\mathcal{X}_0$ snc

$D(\mathcal{X}_0)$ dual complex

Ex: $\mathcal{X} = \{ t f_4 + x_0 x_1 x_2 x_3 = 0 \} \subseteq \mathbb{P}^3_x \times \mathbb{C}_t$

$\mathcal{X}_0 = (x_0 x_1 x_2 x_3 = 0) \subseteq \mathbb{P}^3_{\mathbb{C}}$



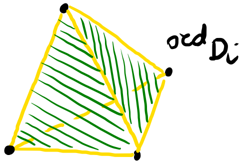


- X^{an} Berkovich space of X
  $\supseteq \{ v : K(X)^{\times} \longrightarrow \mathbb{R} \mid v(t) = 1 \}$
  $v(ab) = v(a) + v(b)$
  $v(a+b) \geq \min\{ v(a), v(b) \}$

- $\mathcal{X}_0 = \sum D_i$ snc degeneration of X
  $D_i \longmapsto \mathrm{ord}_{D_i}$ order of vanishing
  $\overset{\shortparallel}{(f_i = 0)}$ $\quad \mathrm{ord}_{D_i}(f) = \mathrm{ord}_{D_i}(f_i^a g) = a$

- embedding $D(\mathcal{X}_0) \hookrightarrow X^{an}$

$Sk(\mathcal{X})$ skeleton of $\mathcal{X}$



- retraction $\rho_{\mathcal{X}} : X^{an} \longrightarrow Sk(\mathcal{X})$

Integral affine structure on B : via non-archimedean SYZ fibration

$X \longrightarrow \Delta^* \subseteq \mathbb{C}_t$

$\mathcal{X} \longrightarrow \Delta \subseteq \mathbb{C}_t$

$\mathcal{X}$ good minimal dlt degeneration

$\|$

good : comp's of $\mathcal{X}_0$ are Cartier divisor of $\mathcal{X}$

min : $K_{\mathcal{X}} + \mathcal{X}_0 \sim 0$

dlt : generalization of $(\mathcal{X}, \mathcal{X}_0)$ snc

$D(\mathcal{X}) \xhookrightarrow{\sim} Sk(\mathcal{X}) = Sk^{ess}(X)$

essential skeleton

[Nicaise - Xu - Yu] $\rho_{\mathcal{X}} : X^{an} \longrightarrow Sk^{ess}(X)$ is an affinoid torus fibration away from a locus Γ of codim 2

$$\begin{array}{ccccc} X^{an} & & (\mathbb{G}_m^n)^{an} & & v \\ \downarrow \rho & \text{locally isomorphic} & \downarrow \text{trop} & & \downarrow \\ Sk^{ess}(X) & & \mathbb{R}^n & & v(z_i) \end{array}$$

In particular : ρ induces an integral affine structure on $Sk^{ess}(X) \setminus \Gamma$

Main results

$\mathcal{X} \longrightarrow \Delta^* \subseteq \mathbb{C}_t$

$\mathcal{X} \longrightarrow \Delta \subseteq \mathbb{C}_t$ good min dlt degeneration

SYZ conjecture

$$\begin{array}{ccc} X_t & & X^{an} \\ \big\downarrow \pi & & \big\downarrow \rho_{\mathcal{X}} \\ B & = D(\mathcal{X}) \simeq & Sk^{ess}(X) \end{array}$$

non-arch SYZ fibration

essential skeleton

[Brown-Mazzon] Let X be birational to $Hilb^n(S)$ or $K_n(A)$ (families of CY of dim $2n$) where S K3 surface, A abelian surface $/\Delta^*$ and max degenerate. Then $Sk^{ess}(X)$ is homeomorphic to $\mathbb{CP}^n$

$V(f_4) \subseteq \mathbb{P}^3$

$V(f_5) \subseteq \mathbb{P}^4$

[Mazzon-Pille-Schneider] in preparation

For degenerations of quartic K3 surfaces and quintic 3-folds by non-archimedean SYZ fibration, $Sk^{ess}(X) \simeq S^n$ can be endowed with an integral affine structure equal to the one classically constructed on B in mirror symmetry