Punctual Hilbert and Quot schemes on Kleinian singularities

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Outline

1. Hilbert scheme of points on curves and smooth surfaces
2. The orbifold and coarse Hilbert schemes
3. Euler characteristic calculations (joint work with András Némethi and Balázs Szendrői)
4. Quiver variety description (joint work with Alastair Craw, Søren Gammelgaard and Balázs Szendrői)
5. Quot schemes (work in progress)

Hilbert scheme of points

Let X be a quasiprojective variety over $\mathbb{C}$.

Definition (Theorem)

For every $n \in \mathbb{N}$ there is a Hilbert scheme $\mathrm{Hilb}^n(X)$, which parametrizes 0 dimensional subschemes (ideal sheaves) of colength n on X.

Remark

1. $\mathrm{Hilb}^n(X)$ *represents a moduli functor.*
2. *Every* $Z \in \mathrm{Hilb}^n(X)$ *decomposes as* $Z = \coprod Z_j$, *where the supports* $P_j = \mathrm{supp}(Z_j)$ *are mutually disjoint.*
3. $\mathrm{colength}(Z, P_j) = h^0(X, \mathcal{O}_{Z,P_j})$.
4. $n = \mathrm{colength}(Z) = \sum_j \mathrm{colength}(Z, P_j)$.
5. *Hilbert-Chow morphism*

$$\Pi : \mathrm{Hilb}^n(X) \to S^n X, \quad Z \mapsto \sum_j \mathrm{colength}(Z, P_j) P_j.$$

Hilbert scheme of points

Question: From topological/analytical properties of X what can we infere about the topology of $\mathrm{Hilb}^n(X)$?

- Usually better to work with the collection of the Hilbert schemes for all n together.

$$Z_X(q) = \sum_{n=0}^{\infty} \chi_{\mathrm{top}}(\mathrm{Hilb}^n(X))q^n$$

- Relative version: For $Y \subset X$ a (locally) closed subvariety: $\mathrm{Hilb}^n(X, Y) \subset \mathrm{Hilb}^n(X)$ is the Hilbert scheme of points (set-theoretically) supported on Y .

Curves

C curve over $\mathbb{C}$ with singularities $p_i \rightsquigarrow C_{sm} = C \setminus \coprod_i p_i$ smooth part

$$\begin{aligned} Z_C(q) &= \sum_{n=0}^{\infty} \chi(\mathrm{Hilb}^n(C))q^n \\ &= \sum_{n=0}^{\infty} \sum_{n_0+\cdots=n} \chi(\mathrm{Hilb}^{n_0}(C_{sm}))q^{n_0} \prod_i \chi(\mathrm{Hilb}^{n_i}(C,p_i))q^{n_i} \\ &= Z_{C_{sm}}(q) \prod_i \underbrace{\left(\sum_{n=0}^{\infty} \chi(\mathrm{Hilb}^n(C,p_i))q^n\right)}_{Z_{(C,p_i)}(q)} \end{aligned}$$

Theorem (Macdonald, Maulik, Oblomkov-Shende)

1. $\Pi : \mathrm{Hilb}^n(C_{sm}) \to S^n C_{sm}$ *is an isomorphism*
2. $Z_{C_{sm}}(q) = \frac{1}{(1-q)^{\chi(C_{sm})}}$
3. *If $p \in C$ is a planar singularity, then the topology of the link $L_{(C,0)} \subseteq S^3$ (i.e. its embedding type) determines $Z_{(C,p)}(q)$*

Smooth surfaces

X surface over $\mathbb{C}$ with *isolated* singularities $p_i \rightsquigarrow X_{sm} = X \setminus \coprod_i p_i$ smooth part.

Again,

$$Z_X(q) = Z_{X_{sm}}(q) \prod_i Z_{(X,p_i)}(q).$$

Theorem (Fogarty, Ellingsrud-Strømme, Göttsche)

1. $\mathrm{Hilb}^n(X_{sm})$ *is smooth of dimension* $2n$
2. $\Pi : \mathrm{Hilb}^n(X_{sm}) \to S^n X_{sm}$ *is a resolution of singularities*
3. $$Z_{X_{sm}}(q) = \prod_{i=1}^{\infty} \frac{1}{(1-q^n)^{\chi(X_{sm})}}$$
4. *In particular, if* $X = X_{sm}$ *is smooth, then up to a suitable power of* q, $Z_X(q)$ *transforms a modular form.*

An example: the affine plane

Let $X = \mathbb{C}^2$.

- Points of $\mathrm{Hilb}^n(\mathbb{C}^2) \leftrightarrow$ Ideals in $\mathbb{C}[x, y]$ of colength n.
- Special ideals: monomial ideals $\leftrightarrow$ partitions.
- Using the technique of torus localization, we obtain

$$\begin{aligned} \chi(\mathrm{Hilb}^n(\mathbb{C}^2)) &= \#\{\text{monomial ideals of colength } n\} \\ &= \#\{\lambda \text{ a partition of } n\} \\ &= p(n) \end{aligned}$$

- and so

$$Z_{\mathbb{C}^2}(q) = \sum_{n \geq 0} p(n) q^n = \prod_{i=1}^{\infty} \frac{1}{1 - q^n}$$

as stated by Göttsche's formula.

An example: the affine plane (cont'd)

$$Z_{\mathbb{C}^2}(q) = \prod_{i=1}^{\infty} \frac{1}{1-q^n} = \sum_{n\geq 0} p(n) q^n \ .$$

This is the character formula for $\mathcal{F}$, the Fock space representation of the Heisenberg algebra.
Recall:

- $\mathcal{F} = \bigoplus_{\lambda \in \mathcal{P}} \mathbb{C}\lambda$,
- $p(\lambda) = \sum_{1 \text{ block added}} \lambda'$,
- $q(\lambda) = \sum_{1 \text{ block removed}} \lambda'$,
- $[p, q] = Id$.

Theorem (Nakajima, Grojnowski)

$$H^*(\mathrm{Hilb}(\mathbb{C}^2)) = \bigoplus_{n=0}^{\infty} H^*(\mathrm{Hilb}^n(\mathbb{C}^2)) \cong \mathcal{F} \ .$$

Moreover, the operators p and q can be constructed geometrically.

Hilbert schemes of surface singularities

We study the simplest possible class: *rational double points*.

These can be obtained as quotients:
$G \subset \mathrm{Sl}_2(\mathbb{C})$ finite subgroup, acting on $\mathbb{C}^2$.
$\mathbb{C}^2/G$ qoutient variety.

Definition

Coarse (invariant) Hilbert scheme:

$$\mathrm{Hilb}(\mathbb{C}^2/G) = \{Z \lhd \mathbb{C}[x,y]^G \mid Z \text{ is of finite colength}\} .$$

As before, this decomposes as

$$\mathrm{Hilb}(\mathbb{C}^2/G) = \coprod_{m \in \mathbb{N}} \mathrm{Hilb}^m(\mathbb{C}^2/G) .$$

Orbifold Hilbert schemes

Orbifold singularities have another type of Hilbert scheme.

Definition

Orbifold (equivariant) Hilbert scheme:

$$\mathrm{Hilb}([\mathbb{C}^2/G]) = \{I \in \mathrm{Hilb}(\mathbb{C}^2) \mid I \text{ is } G\text{-invariant}\} .$$

This stratifies as

$$\mathrm{Hilb}([\mathbb{C}^2/G]) = \bigcup_{\rho \in \mathrm{Rep}(G)} \mathrm{Hilb}^{\rho}([\mathbb{C}^2/G]) ,$$

where

$$\mathrm{Hilb}^{\rho}([\mathbb{C}^2/G]) = \{I \in \mathrm{Hilb}(\mathbb{C}^2) \mid H^0(\mathcal{O}_I) = H^0(\mathcal{O}_{\mathbb{C}^2}/I) \simeq_G \rho\} .$$

Scheme-theoretic pushforward map:

$$p_* : \mathrm{Hilb}([\mathbb{C}^2/G]) \to \mathrm{Hilb}(\mathbb{C}^2/G), \quad I \mapsto I^G = I \cap \mathbb{C}[x,y]^G .$$

Generating series

Let $\rho_0, \dots \rho_n$ be the irreducible representations of G.

Definition

(a) Coarse generating series (or coarse partition function):

$$Z_{\mathbb{C}^2/G}(q) = \sum_{m=0}^{\infty} \chi\left(\mathrm{Hilb}^m(\mathbb{C}^2/G)\right) q^m \ .$$

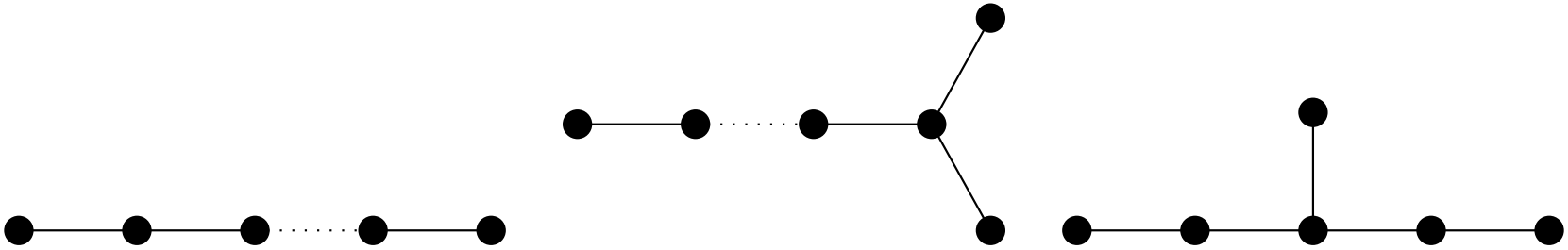
(b) Orbifold generating series (or orbifold partition function):

$$Z_{[\mathbb{C}^2/G]}(q_0, \dots, q_n) =$$

$$= \sum_{m_0,\dots,m_n=0}^{\infty} \chi\left(\mathrm{Hilb}^{m_0\rho_0+\dots+m_n\rho_n}([\mathbb{C}^2/G])\right) q_0^{m_0} \cdot \dots \cdot q_n^{m_n} \ .$$

Simple singularities

Finite subgroups of $\mathrm{Sl}_2(\mathbb{C})$: $A_n (n \geq 1)$, $D_n (n \geq 4)$, E_6, E_7, E_8.
To the quotient $\mathbb{C}^2/G$ we can associate its resolution graph:



There is also a corresponding Lie algebra.

The abelian case

G = cyclic subgroup of $\mathrm{Sl}_2(\mathbb{C})$ of order $n+1$.
Generated by

$$\sigma = \begin{pmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{pmatrix},$$

where ω is a $(n+1)$-st root of unity.

All representations are one dimensional.
They are given by $\rho_j \colon \sigma \mapsto \omega^j$, for $j \in \{0, \ldots, n\}$.

The abelian case: $(\mathbb{C}^*)^2$ fixpoints

Monomials transform according to irreducible representations of G:

$\vdots$							
0	1						
1	2						
$\vdots$	$\vdots$						
n	0		$n-2$	$n-1$	n	0	
0	1	$\cdots$	$n-1$	n	0	1	$\cdots$

We can apply again torus localization to count only the monomial ideals. We get a coloured version of the partition counting problem:

$$\chi\left(\mathrm{Hilb}^{\sum m_i \rho_i}(\mathbb{C}^2)\right) =$$

$$\#\{\lambda \text{ a coloured partition with } m_i \text{ boxes of colour } i\}$$

and so

$$Z_{[\mathbb{C}^2/G]}(q_0, \ldots, q_n) = 1 + \sum_{\lambda} \prod_{j} q_j^{\mathrm{col}_j(\lambda)}$$

is the *coloured generating function of partitions*.

The coarse Hilbert scheme

Points of $\mathrm{Hilb}(\mathbb{C}^2/G) \leftrightarrow$ ideals in $\mathbb{C}[x,y]^G$

Again, we can apply torus localization.

Torus fixed points of $\mathrm{Hilb}(\mathbb{C}^2/G)$
$\leftrightarrow$ monomial ideals in $\mathbb{C}[x,y]^G$
$\leftrightarrow$ Young diagrams where all the generators are of color 0:
0-generated Young diagrams

Colength of a monomial ideal $I \triangleleft \mathbb{C}[x,y]^G =$ number of 0 boxes

In particular,

$$\chi\left(\mathrm{Hilb}^n(\mathbb{C}^2)\right) =$$
$$\#\{\lambda \text{ a 0-generated Young diagram with } n \text{ boxes of colour } 0\}$$

Abelian case cont'd

$\mathbb{C}[x,y]^G = \mathbb{C}[x^{n+1}, xy, y^{n+1}]$

Recall the pushforward map:

$$\mathrm{Hilb}([\mathbb{C}^2/G]) \to \mathrm{Hilb}(\mathbb{C}^2/G),\ I \mapsto I^G = I \cap \mathbb{C}[x,y]^G.$$

On the level of diagrams, we extend the diagram of I to the smallest 0-generated diagram which covers it.

Example

2			
0	1		
1	2		
2	0		
0	1	2	0

Abacus on $n = 3$ rulers: to a diagram λ associate the infinite series of numbers: $\{\lambda_1, \lambda_2 - 1, \lambda_3 - 2, \dots\}$:

$$\begin{array}{ccc}
\vdots & \vdots & \vdots \\
-8 & -7 & -6 \\
-5 & -4 & -3 \\
-2 & -1 & 0 \\
1 & 2 & 3 \\
4 & 5 & 6 \\
\vdots & \vdots & \vdots
\end{array}$$

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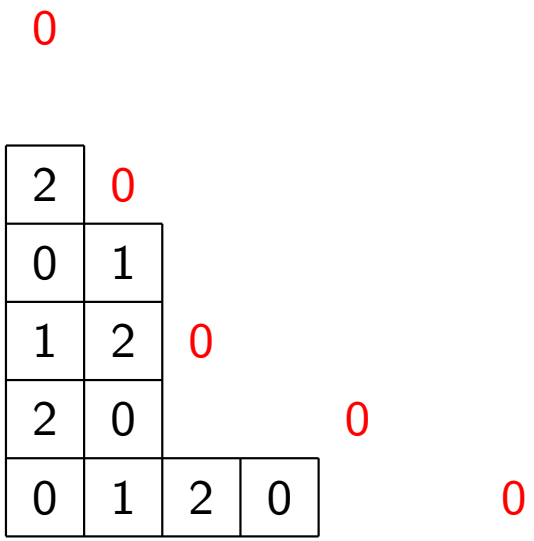

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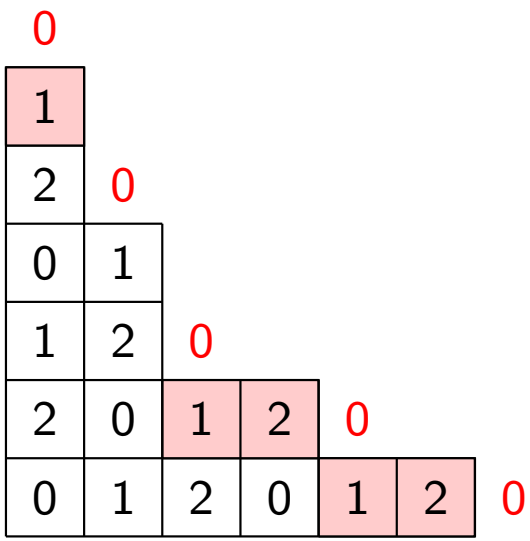
Example



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Abelian case cont'd

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Recall the pushforward map:

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Example

1					
2					
0	1				
1	2				
2	0	1	2		
0	1	2	0	1	2

Abacus on $n = 3$ rulers: to a diagram λ associate the infinite series of numbers: $\{\lambda_1, \lambda_2 - 1, \lambda_3 - 2, \dots\}$:

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\vdots & \vdots & \vdots
\end{array}$$

Abelian case cont'd

So p_* on the abacus corresponds to pushing all the bead to the right as much as possible. In particular,

0-generated Young diagrams	$\longleftrightarrow$	abacus configurations where no bead can be moved to the right.

Theorem (Gy–Némethi–Szendrői, 2015)

- *Let $\xi = \mathrm{e}^{\frac{2\pi i}{n+2}}$. Under the substitution $q_i = \xi$ $i \in \{1, \ldots, n\}$, $q_0 = \xi^{-n} q$ the terms in $Z_{[\mathbb{C}^2/G]}(q_0, \ldots, q_n)$ corresponding to the preimage of a 0-generated diagram λ_0 add up to the term corresponding to λ_0 in $Z_{\mathbb{C}^2/G}(q)$:*

$$\sum_{\mu \in p_*^{-1}(\lambda_0)} \prod_i q_i^{\mathrm{wt_i}(\mu)} \Big|_{q_1 = \cdots = q_n = \xi, q_0 = \xi^{-n} q} = q^{\mathrm{wt_0}(\lambda_0)}.$$

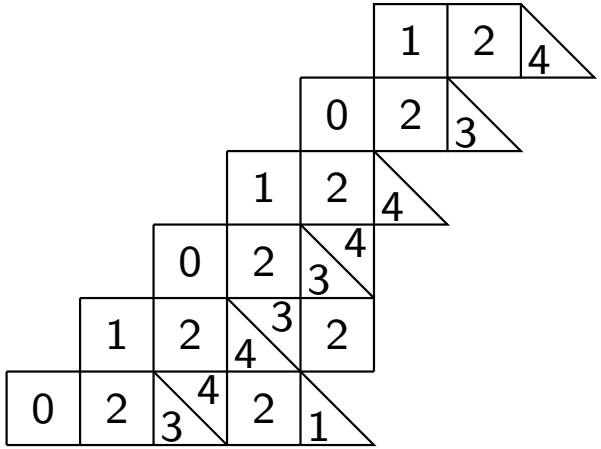
- *As a consequence,*

$$Z_{\mathbb{C}^2/G_\Delta}(q) = Z_{[\mathbb{C}^2/G]}(q_0, \ldots, q_n) \Big|_{q_1 = \cdots = q_n = \xi, q_0 = \xi^{-n} q}.$$

Nonabelian cases

In type D a similar but more complicated combinatorics works. Diagonally colored Young diagrams are replaced by objects called Young walls.

Example ($G = D_4$)



$w_i(\lambda)$ = number of blocks (squares or triangles) with label i.

The substitution works with the root of unity $\zeta = e^{\frac{2\pi i}{1+h}}$, where h is the Coxeter number.

There is no theory of Young diagrams/walls for type E.

The generating functions

Theorem (Nakajima, 90ies)

$\mathrm{Hilb}([\mathbb{C}^2/G])$ *is a smooth quiver variety associated with the corresponding affine Dynkin quiver.* $H^*(\mathrm{Hilb}([\mathbb{C}^2/G])$ *carries an action of the corresponding affine Lie algebra.*

Corollary (using the Weyl–Kac character formula)

$$Z_{[\mathbb{C}^2/G]}(q_0,\ldots,q_n) = \frac{\sum_{\overline{m}=(m_1,\ldots,m_n)\in\mathbb{Z}^n} q_1^{m_1}\cdot\cdots\cdot q_n^{m_n}(q^{1/2})^{\overline{m}^\top\cdot C_\Delta\cdot\overline{m}}}{\prod_{m=1}^{\infty}(1-q^m)^{n+1}}$$

where $q=\prod_{i=0}^n q_i^{d_i}$, $d_i=\dim\rho_i$, *and* C *is the corresponding finite type Cartan matrix.*

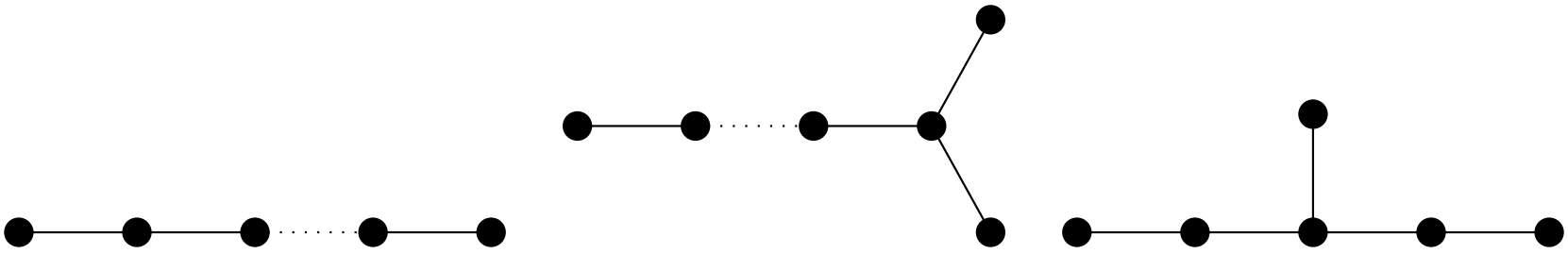
Corollary (Previous Corollary + GyNSz)

For G *of type* A *or* D *(conjecturally of type* E*), let* $\zeta=e^{\frac{2\pi i}{1+h}}$.

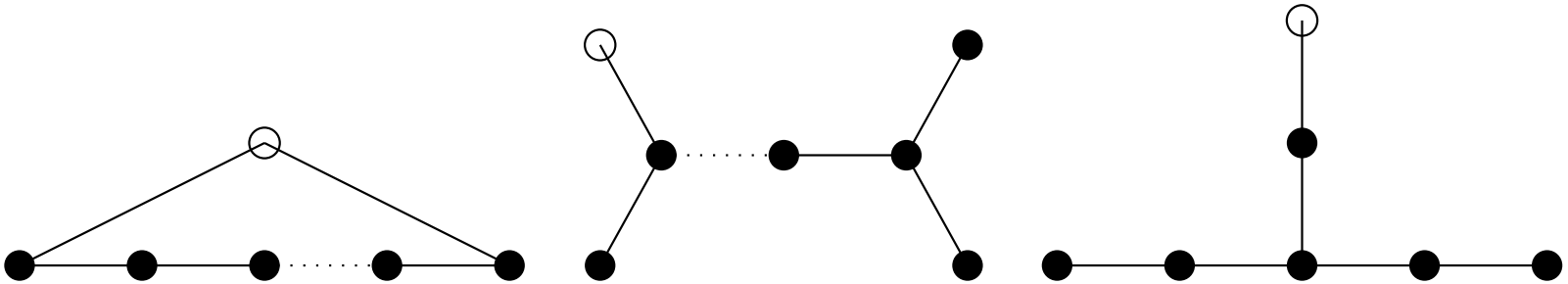
$$Z_{\mathbb{C}^2/G_\Delta}(q) = \frac{\sum_{\overline{m}=(m_1,\ldots,m_n)\in\mathbb{Z}^n} \zeta^{m_1+m_2+\cdots+m_n}(q^{1/2})^{\overline{m}^\top\cdot C_\Delta\cdot\overline{m}}}{\prod_{m=1}^{\infty}(1-q^m)^{n+1}}$$

Simple singularities - McKay correspondence (again)

Finite subgroups of $\mathrm{Sl}_2(\mathbb{C})$: $A_n (n \geq 1)$, $D_n (n \geq 4)$, E_6, E_7, E_8.
To the quotient $\mathbb{C}^2/G$ we can associate its resolution graph:



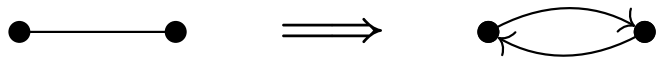
The (irred.) representations of the finite group are described by another graph (McKay quiver, $\rho_{\mathrm{def}} \otimes \rho_i = \bigoplus_j a_{ij}\rho_j$):



Doubled framed McKay quiver

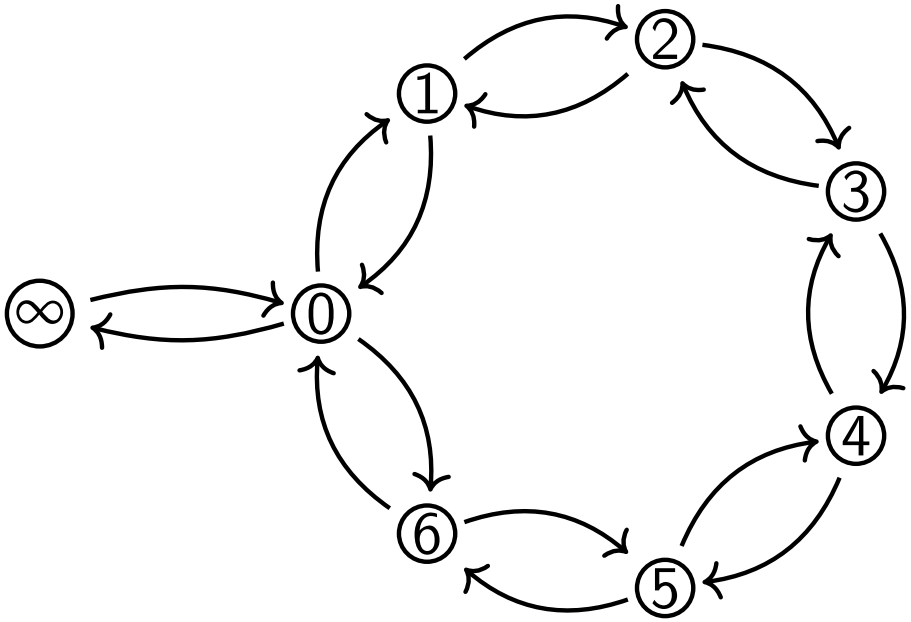
$Q = (Q_0, Q_1)$: starting from the McKay quiver

- add one framing vertex at 0: $Q_0 = \{\infty, 0, 1, \ldots, n\}$,
- double the arrows: Q_1.



Example

$G = \mathbb{Z}_7 = A_6$



Moduli space of quiver representations

Dimension vector: $v := (v_i)_{i \in Q_0} \in \mathbb{N}^{Q_0}$ with $v_\infty = 1$.
The space of stability conditions:

$$\Theta_v := \{\theta : \mathbb{Z}^{Q_0} \to \mathbb{Q} \,|\, \theta(v) = 0\}$$

The space of representations:

$$\mathrm{Rep}(Q, v) := \oplus_{a \in Q_1} \mathrm{Hom}(\mathbb{C}^{v_{t(a)}}, \mathbb{C}^{v_{h(a)}})$$

Symmetry group: $G(v) = \prod_{0 \le i \le n} GL(v_i, \mathbb{C})$.
Let $G := G(v)/(\text{diagonal } \mathbb{C}^*)$.
Its action on $\mathrm{Rep}(Q, v)$ induces a moment map:

$$\mu : \mathrm{Rep}(Q, v) \to \mathfrak{g}^*$$

Moduli space of quiver representations

Any $\theta \in \Theta_v$ induces a character:

$$\begin{aligned} \chi_\theta(g) : G &\to \mathbb{C}^* \\ g &\mapsto \prod_{i \in Q_0} \det(g_i)^{-\theta_i} \end{aligned}$$

A representation M is θ-stable (respectively semistable) if $\theta(\dim M) = 0$ and for every proper, nonzero subrepresentation $N \subset M$, we have $\theta(\dim N) > 0$ (respectively $\theta(\dim N) \geq 0$).

Definition (Nakajima quiver variety)

For any $\theta \in \Theta_v$, let

$$\mathfrak{M}_\theta(v) := \mu^{-1}(0) /\!\!/_\theta G$$

be the GIT quotient of the locus of θ-semistable points of $\mu^{-1}(0)$ by the action of G.

Quiver varieties - examples

Example (Kronheimer–Nakajima)

Choose $v_1 = v_{reg} = \{\dim \rho_i\}$. Then for generic stability condition θ, the GIT quotient $\mathfrak{M}_\theta(v_1)$ is independent of θ, and is isomorphic to the minimal resolution Y of the surface singularity $X = \mathbb{C}^2/G$.

Example (folklore)

Let $v_1 = n \cdot v_{reg} = \{n \cdot \dim \rho_i\}$ for some natural number n. Choose the stability condition $\theta = 0$. Then the GIT quotient $\mathfrak{M}_0(v_n)$ is affine (general fact), and is isomorphic to the n-th symmetric product $S^n(X)$. In particular, $\mathfrak{M}_0(v_n) \cong X$.

Generic and special stability parameters

We continue to work with this setup: fix $v_n = n \cdot v_{reg} = \{n \cdot \dim \rho_i\}$, and study the space $\mathfrak{M}_\theta(v_n)$ as the stability parameter θ varies.

By general principles of variation of GIT (Thaddeus, Dolgachev-Hu), one expects a wall-and-chamber structure, with stability parameters in open chambers giving nice GIT quotients $\mathfrak{M}_\theta(v_n)$, while the quotient $\mathfrak{M}_{\theta_0}(v_n)$ becomes more singular for parameters θ_0 lying in walls.

The general setup will also induce morphisms

$$\mathfrak{M}_\theta(v_n) \to \mathfrak{M}_{\theta_0}(v_n)$$

relating different quiver varieties.

Example (continued)

With $v_1 = \{\dim \rho_i\}$ as above, moving from a generic stability condition θ to $\theta = 0$ gives a morphism $\mathfrak{M}_\theta(v_1) \to \mathfrak{M}_0(v_1)$ which can be identified with the minimal resolution $Y \to X = \mathbb{C}^2/G$.

A distinguished chamber in stability space

Theorem (Varagnolo–Vasserot, Kuznetsov)

Fix $n \geq 1$. There exists a distinguished open chamber $C^+ \subset \Theta$ inside stability space, so that for $\theta \in C^+$,

$$\mathfrak{M}_\theta(v_n) \cong \mathrm{Hilb}^{n \cdot \rho_{\mathrm{reg}}}(\mathbb{C}^2)$$

where on the right we have the G-equivariant Hilbert scheme of $\mathbb{C}^2$ corresponding to $n \cdot \rho_{\mathrm{reg}} \in \mathrm{Rep}(G)$, with ρ_{reg} the regular representation.

The morphism to the zero of the stability space can be identified with

$$\mathrm{Hilb}^{n \cdot \rho_{\mathrm{reg}}}(\mathbb{C}^2) \to S^n(\mathbb{C}^2/\Gamma)$$

which is a minimal resolution of singularities.

Example (continued again, thm of Kapranov–Vasserot)

For $n = 1$, $\mathrm{Hilb}^{\rho_{\mathrm{reg}}}(\mathbb{C}^2) \cong Y$, the minimal resolution of $\mathbb{C}^2/\Gamma$.

The wall-and-chamber structure of stability space

Theorem (Bellamy–Craw)

1. Θ *has a wall-and-chamber structure described combinatorially by the affine root system.*
2. *The simplicial cone*

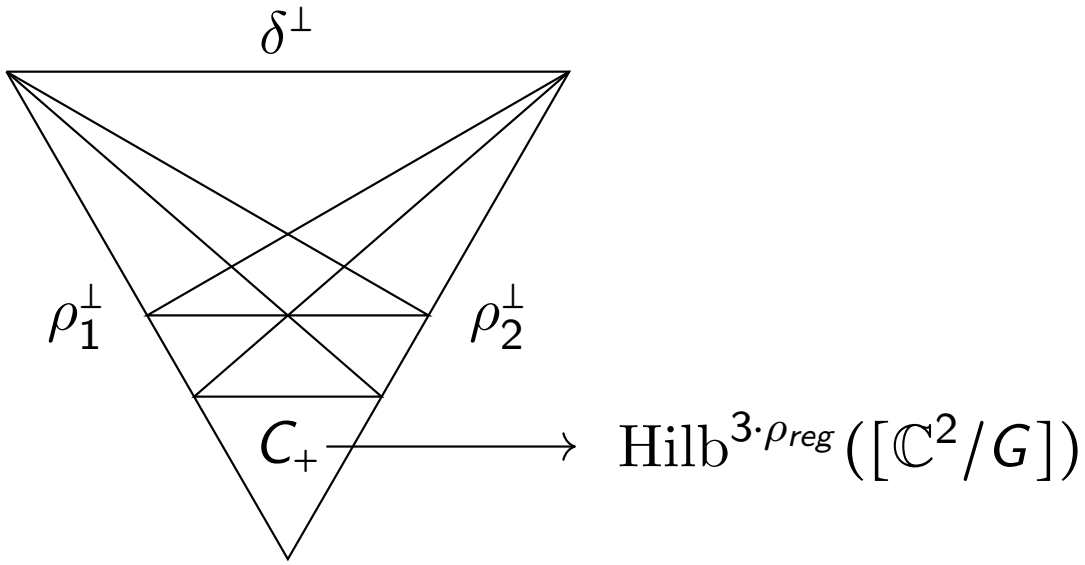
$$F := \langle \delta, \rho_1, \ldots, \rho_n \rangle^\vee \subset \Theta$$

   is a fundamental domain for the Weyl group action on Θ.
3. *There is a closed cone* $\bar{C}^+$ *that can be identified with the nef cone (closed ample cone) of the variety* $\mathrm{Hilb}^{n \cdot \rho_{\mathrm{reg}}}(\mathbb{C}^2)$.
4. *Every open chamber of* F *corresponds to the ample cone of a birational model of* $\mathrm{Hilb}^{n \cdot \rho_{\mathrm{reg}}}(\mathbb{C}^2)$.

Space of stability conditions

Example

The height 1 transversal slice of $F = \langle \delta, \rho_1, \rho_2 \rangle^\vee$ for A_2 and $\rho = 3\rho_{reg}$:



A distinguished corner of the stability space

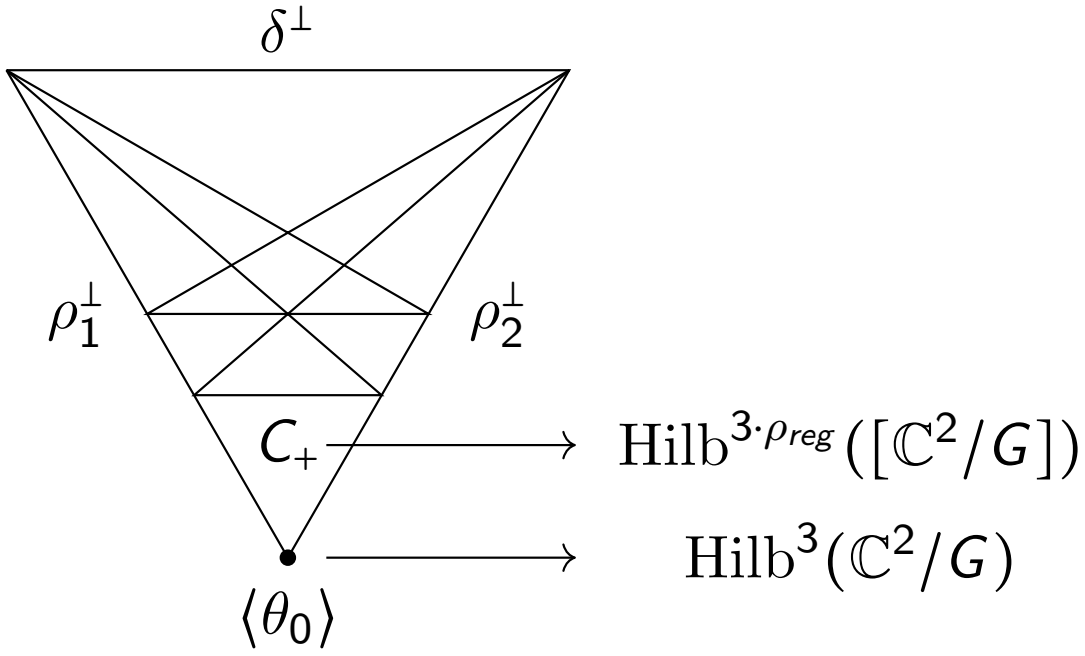
Theorem (Craw–Gammelgaard–Gy–Szendrői, 2019)

For a distinguished ray $\theta_0 \in \partial \bar{C}^+$, we have an isomorphism

$$\mathfrak{M}_{\theta_0}(v_n) \cong \mathrm{Hilb}^n(\mathbb{C}^2)$$

between a quiver variety and the Hilbert scheme of points of the surface singularity.

Example



A distinguished corner of stability space (continued)

Theorem (continued)

The resulting chain of morphisms

$$\mathfrak{M}_\theta(v_n) \to \mathfrak{M}_{\theta_0}(v_n) \to \mathfrak{M}_0(v_n)$$

can be identified with the chain

$$\mathrm{Hilb}^{n\cdot\rho_{\mathrm{reg}}}(\mathbb{C}^2) \to \mathrm{Hilb}^n(\mathbb{C}^2/G) \to S^n(\mathbb{C}^2/G)$$

which includes the Hilbert–Chow morphism of the variety $\mathbb{C}^2/G$*.*

Corollary

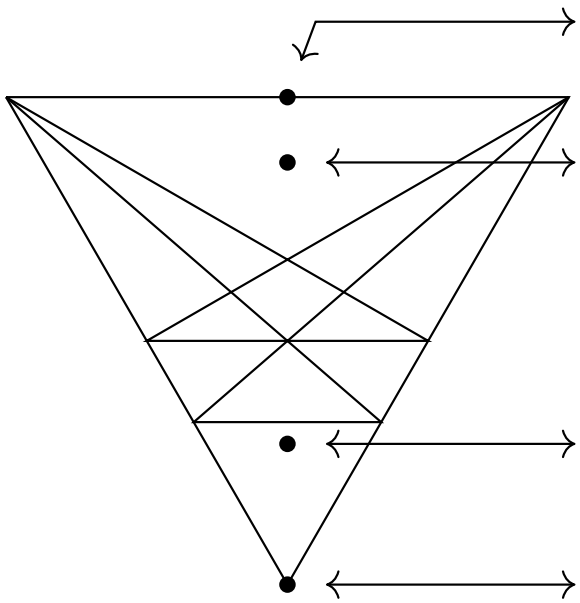
The Hilbert scheme $\mathrm{Hilb}^n(X)$ *of the surface singularity* $X = \mathbb{C}^2/G$ *is an irreducible, normal quasiprojective variety with a unique symplectic (Calabi–Yau) resolution.*

This is about as nice as one could hope for! Conjecturally this property *characterises* surface rational double points among all varieties of dimension at least 2.

Many spaces in one diagram

We get the following diagram of GIT-induced morphisms, including the Hilbert–Chow morphisms of both the singularity $X = \mathbb{C}^2/G$ and its minimal resolution Y.

Example



$$\begin{tikzcd}
 & S^n(Y) \arrow[dr] & \\
 & \mathrm{Hilb}^n(Y) \arrow[u] \arrow[r] \arrow[d, dashed, "\cong"] & S^n(\mathbb{C}^2/G) \\
 & \mathrm{Hilb}^{n\cdot\rho_{reg}}([\mathbb{C}^2/G]) \arrow[ur] \arrow[d] & \\
 & \mathrm{Hilb}^n(\mathbb{C}^2/G) \arrow[uur] &
\end{tikzcd}$$

Back to Euler characteristics

We have identified the resolution of singularities

$$\mathrm{Hilb}^{n\cdot\rho_{\mathrm{reg}}}(\mathbb{C}^2) \to \mathrm{Hilb}^n(X)$$

with a map

$$\mathfrak{M}_\theta(v_n) \to \mathfrak{M}_{\theta_0}(v_n)$$

between quiver varieties.

This suggests that the conjecture of Gy–Némethi–Szendrői about the generating function of Euler characteristics of $\mathrm{Hilb}^n(X)$ could be approached this way.

Nakajima: the fibres of the map $\mathfrak{M}_\theta(v_n) \to \mathfrak{M}_{\theta_0}(v_n)$ between quiver varieties are themselves (Lagrangian subvarieties in) quiver varieties associated with *finite ADE quivers*.

This looks like it gives an approach to the conjecture. However, computing the Euler characteristics of fibres directly is still hard! Nevertheless, using his earlier results based on supercomputer calculations...

Theorem (Nakajima, 2020)

For G of arbitrary type, *with $q = \prod_i q_i^{\delta_i}$ and $\xi = \exp(\frac{2\pi i}{1+h})$, the generating function of the Hilbert scheme of points of the surface singularity $X = \mathbb{C}^2/G$ is related to the equivariant generating function by the formula where h is the Coxeter number of the Lie algebra of the corresponding type. In other words, the conjecture of Gy–Némethi–Szendrői from 2015 holds.*

Further directions (work in progress)

Do other walls in the space of stability parameters correspond to moduli spaces? The answer, at least in type A, seems to be yes.

Proposition

1. *For every $j \in \{0, 1, \ldots, n\}$ there is a ray $\langle \theta_j \rangle \in \partial \bar{C}^+$ such that*

$$\mathfrak{M}_{\theta_j}(v_n) \cong \mathrm{Quot}^n(\mathbb{C}^2/G, \mathcal{R}_j)$$

   where the $\mathcal{R}_j$ are indecomposable reflexive modules on $(\mathbb{C}^2/G, 0)$.

2. *The substitution $q_i = \xi, i \in \{0, 1, \ldots, n\} \setminus \{j\}$, $q_j = \xi^{-n} q$ into $Z_{[\mathbb{C}^2/G]}(q_0, \ldots, q_n)$ gives the generating series*

$$Z_{\mathbb{C}^2/G,j}(q) = \sum_{n=0}^{\infty} \chi\left(\mathrm{Quot}^n(\mathbb{C}^2/G, \mathcal{R}_j)\right) q^m.$$

3. *Combinatorially: count the j-colength of j-generated partitions.*

Thank you for your attention!

Questions?