Rigid, not infinitesimally rigid surfaces with ample canonical bundle

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R. ~~B~~ Pignatelli

Kodaira / Morrow 71: $\exists$? X s.th. ~~BBX~~ $\mathrm{Def}(X)$ = non reduced point. $(*)$

Bauer / Pignatelli '18: $\exists$ S gen'l type

$$S \longrightarrow \bar{S}$$

can. model

smooth
rigid, not inf.-rigid

nodal rigid & inf. rigid

Q: Is it allways like this? (No)

Vakil '06: "Murphy's law in algebraic geometry".

Q: $\not\Rightarrow$ $(*)$?

Vakil's Setup

Kapovich/Millson: 'All X occur as incidence schemes

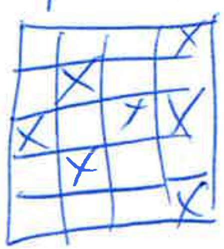
Vakil

Mnëv $\forall X\ \exists n$ $(\mathbb{P}^2)^{\vee} \times (\check{\mathbb{P}}^2)^m$

s.th. $\boxed{X \times \mathbb{A}^N}$ containing a linearrangem.

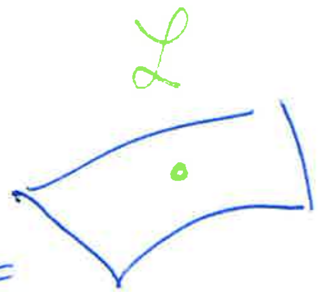
points

lines

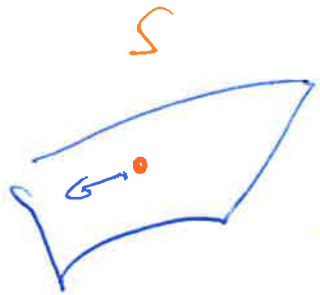


$\cup$

$I(\mathcal{L}) =$



$\cong$? ∂

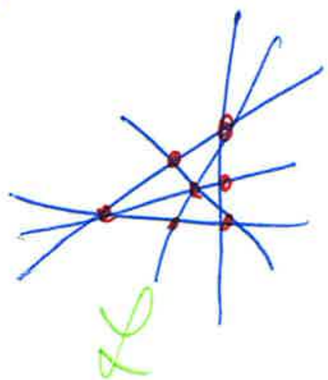


$= \mathrm{Def}(S)$

Racelli.

$\cup$ $\quad$ $\cup$

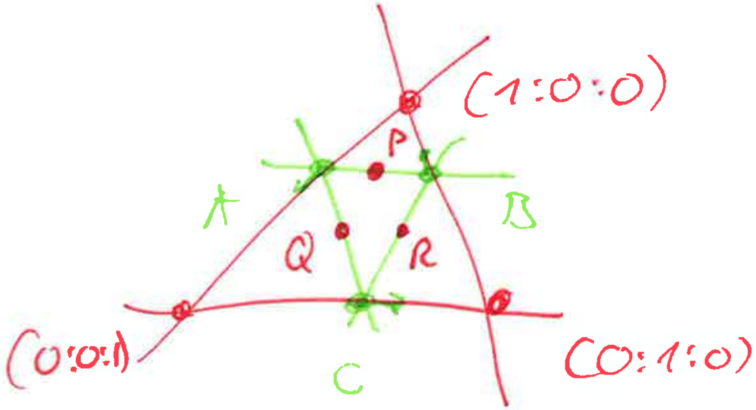
Problem: Not all incidence schemes come from linearrangement. Ex: $x^2=0$



$\leftarrow \mathrm{Bl}(\mathbb{P}^2) \xleftarrow[\text{cover}]{\text{delete}} S$

+ Building data $\nearrow$ Pardini Fantechi.

generalized incidence schemes



Def

given: fixed points & lines
variable points & lines

$$I \subset (\mathbb{P}^2)^{\#\text{v. point}} \times (\check{\mathbb{P}}^2)^{\#\text{v. lines}}$$

Defined by $\quad p \in L$

(0,0)
(0,1)
(1,c)
(1,1)

Here

$$\begin{vmatrix} P_1 & P_2 & P_3 \\ a & 0 & 1 \\ 1 & b & 0 \end{vmatrix} = \left| \quad \right| = \left| \quad \right| = 0$$

↳ 3 quadrics

↳ eliminate a, b ⇒ deg 2 polynomial in c

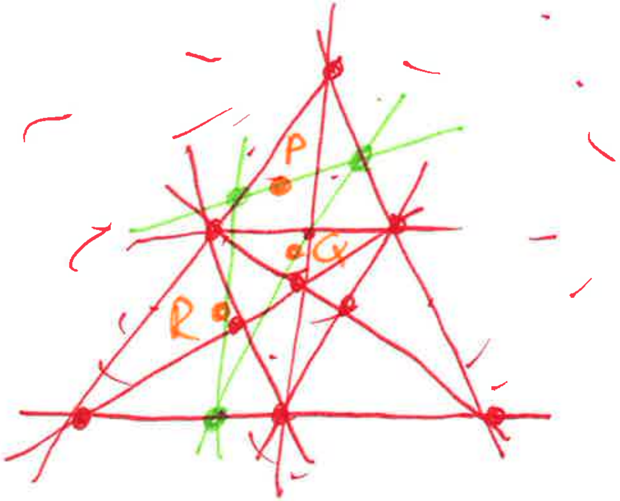
Ex: (2:1:1)
(1:2:1) $\in D$
(1:1:2)

If $D(P,Q,R) = 0$ ⇒ non reduced point

Parchi / Fantechi / Paretti :

g. Incidence Schemes with fixed data

(1:0:0)
(0:1:0)
(0:0:1)
(1:1:1)



Points | Lines

4 ⇒ 6 ⇒ 7 ⇒ 9 ⇒ 13 ⇒ 25 ⇒ 97

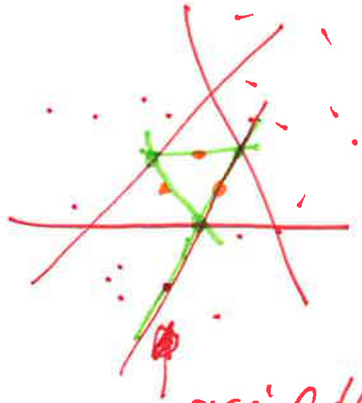
+ 2 lines

P   Q   R

Add L_P, L_Q, L_R, A, B, C

Ist $(P, Q, R) \in D \Longrightarrow$ ☺ ? No

Problem



accidently

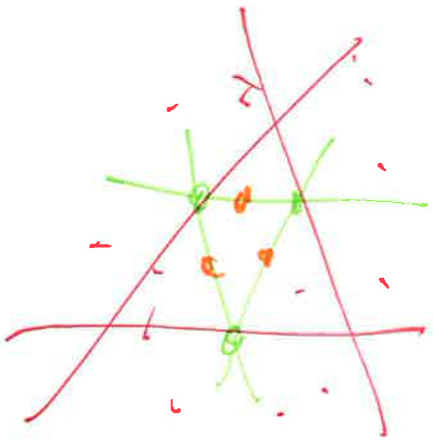
⟹ L_Q is fixed

⟹ rigid & inf. rigid.

This happens every 1.000000 time ☺

Why?

our 200 points.



If $\angle$'s are of small height.

$$\Big|\sum_i l_i P_i\Big| \leq 100$$

choice $1:200$

ht 10 ht 10

$(P,Q,R) \in D$ $(2,2,2)$

$$|D(P,Q,R)| \leq 10^6$$

ht 10

Solution

Observe: D is rational uni

① Find points of medium height on D

② Do (P,Q,R) do lie on 2 fixed lines

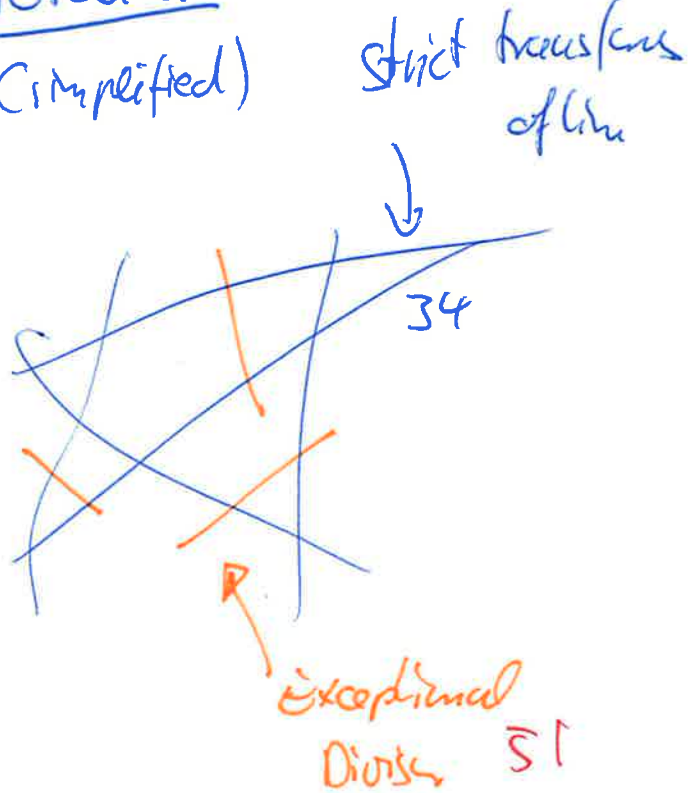
~ 100

$\longrightarrow$ Example with ℓ's not sad.

$(P,Q,R) = (1:4:2),\ (3:14:3),\ (14:25:1)$

Pardini

(simplified)



$Bl(\mathbb{P}^2)$
(in the singular points)

Building data

(each component of branch divisor S_Y

Elements of G:

$G = (\mathbb{Z}/p)^r$

$D_{g_1}, D_{g_2}, \dots, D_{g_{85}}$ (all other $D_g = 0$)

A smooth cover exists:

(1) $\forall \chi \in G^*$

$$\sum_{g\in G} \langle \chi, g\rangle D_g \equiv 0 \mod p \quad \subset \mathrm{Pic}$$

(2) $\langle g_i \rangle \neq \langle g_j \rangle \quad \forall\, i \neq j$

(3) $\langle g_1, \dots, g_{85} \rangle = G$

Find Building Data

$$D = H - \sum a_i E_i$$

$$E = \sum a_i$$

① Choose $g_1 \ldots g_{33} \in (\mathbb{Z}/3)^6$ randomly

② Set $g_{34} = -\sum_{i=1}^{33} g_i$

$\Big\} \Rightarrow$ coefficients of H will be $0 \bmod p$

③ Set

$$g(E) = \sum_{C \cap E = \emptyset} g(C)$$

$\Big\} \Rightarrow$ coefficients of E will be $0 \bmod p$

What about (5): This is the Birthday-Problem

$$\#\text{ lines in } G = \frac{3^6 - 1}{3-1} = 364$$

(c) never a problem

How probable is that 85 People have all different Birthdays?

1:45000 (1. Trial)

$\Rightarrow$ Building data exist.

Nanolfi: Cohomological vanishing conditions
that can be checked on $\mathbb{P}^2$

Problem: Never satisfied in our situation.

Solution: The conditions of Nanolfi can
be weakened (same proof)

$\Rightarrow$ always satisfied (in our example)

K_S is ample: Nakai-Moishezon & intersection number
calculation

$\boxed{p=3}$
$p \geq 2$.

$\Rightarrow \deg L = 1$

$\square$