Fine interior of a lattice polytope: MMP and MS

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What is Fine interior?

$M \cong \mathbb{Z}^d$, $N := \mathrm{Hom}(M, \mathbb{Z})$, $\langle *, * \rangle : M \times N \to \mathbb{Z}$ pairing
$M_{\mathbb{R}} := M \otimes \mathbb{R}$, $N_{\mathbb{R}} := N \otimes \mathbb{R}$ are vector spaces over $\mathbb{R}$.
$A \subset M$ a finite subset, $P := \mathrm{conv}(A) \subset M_{\mathbb{R}}$ a full-dimensional lattice polytope.
e Consider the piecewise linear function $\mathrm{ord}_P : N_{\mathbb{R}} \to \mathbb{R}$:

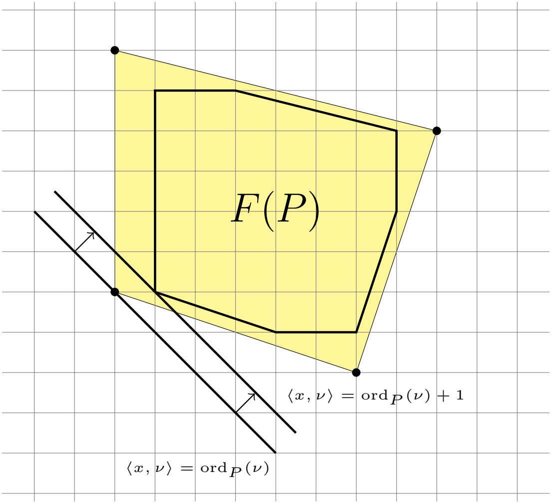
$$\mathrm{ord}_P(y) = \min_{x \in P} \langle x, y \rangle, \quad y \in N_{\mathbb{R}}.$$

Then $P = \{x \in M_{\mathbb{R}} \ : \ \langle x, \nu \rangle \geq \mathrm{ord}_P(\nu) \quad \forall \nu \in N \setminus \{0\}\}$.

Definition

$F(P) := \{x \in M_{\mathbb{R}} \ : \ \langle x, \nu \rangle \geq \mathrm{ord}_P(\nu) + 1 \quad \forall \nu \in N \setminus \{0\}\}$
is called Fine interior of P.

$F(P) = \mathrm{Conv}(P^\circ \cap M)$ for a 2-dimensional lattice polytope



Bibliography

- Jonathan Fine, *Resolution and completion of algebraic varieties*, Ph.D.-Thesis, University of Warwick, June 1983. (in [Ph.D.-Thesis, §4] $F(P)$ is called heart of P.)

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- Jonathan Fine, *Resolution and completion of algebraic varieties*, Ph.D.-Thesis, University of Warwick, June 1983. (in [Ph.D.-Thesis, §4] $F(P)$ is called heart of P.)
- Miles Reid, *Young person's guide to canonical singularities*, Algebraic geometry, Bowdoin, 1985.

Inspirations

- V.V. Danilov, *The geometry of toric varieties*, Russian Math. Surveys **33 :2** (1978), 97-154.
- A.G.Khovanskiĭ, *Newton polyhedra and the genus of complete intersections*, Func. Anal. Appl. **12** (1978), 51-61.

Lattice polytopes P are considered as Newton polytopes of non-degenerate hypersurfaces in the algebraic torus

$$\mathbb{T} = \mathrm{Hom}(M, \mathbb{C}^*).$$

Valuations in birational geometry of algebraic varieties:
M is the lattice of monomials, or characters of $\mathbb{T}$
N is the lattice of valuations of the function field $\mathbb{C}(\mathbb{T})$

What is MMP?

Attempts to find a "good" geometric representatives X in a given birational class of d-dimensional varieties ($d \geq 3$).

Our payment: We have to accept existence mild singularities of "good representatives" X.

Criterion: "Good behavour" with respect to differential forms of top degree (wrt. canonical class K_X),
non-negativity of K_X (if possible).

Discrepancies

X a normal irreducible quasi-projective $\mathbb{Q}$-Gorenstein algebraic variety. Take a resolution of singularities of X

$$\rho : Y \to X$$

with the exceptional locus $\bigcup_{i=1}^{r} D_i$ union of smooth irreducible divisors with only normal crossings.

$I := \{1, \dots, r\}$

$$K_Y = \rho^* K_X + \sum_{i \in I} a_i D_i,$$

The rational numbers $a_i \in \mathbb{Q}$ $(i \in I)$ are called *discrepancies* of divisors D_i.

Singularities in MMP

Definition

Singularities of X are called at worst

- *terminal* if $a_i > 0$, $\forall i \in I$;
- *canonical* if $a_i \geq 0$, $\forall i \in I$;
- *log-terminal* if $a_i > -1$, $\forall i \in I$.

Minimal model

Definition

A normal projective variety X with at worst $\mathbb{Q}$-factorial terminal singularities is called *minimal model* if the canonical class K_X is a semi-ample $\mathbb{Q}$-Cartier divisor.

Non-degenerate hypersurfaces

Definition

A Laurent polynomial

$$f(\mathbf{t}) = \sum_{m \in A} a_m \mathbf{t}^m \in \mathbb{C}[M] \cong \mathbb{C}[t_1^{\pm 1}, \dots, t_d^{\pm 1}]$$

with Newton polytope $P = \operatorname{conv}(A) \subset M_{\mathbb{R}} := M \otimes \mathbb{R}$ is called *non-degenerate* if for any face $P' \preceq P$ the affine hypersurface

$$Z_{f,P'} := \{ \sum_{m \in A \cap P'} a_m \mathbf{t}^m = 0 \} \subset \mathbb{T}.$$

is smooth. The non-degeneracy of $f(\mathbf{t})$ is a Zariski open condition on its coefficients $\{a_m\} \in \mathbb{C}^{|A \cap M|}$.

Theorem of Ishii: Existence of a minimal model

- Shihoko Ishii, *The minimal model theorem for divisors of toric varieties*, Tohoku Math. J. (1999), 213-226.

Theorem of Ishii

A non-degenerate hypersurface $Z \subset \mathbb{T}$ defined by a Laurent polynomial $f \in \mathbb{C}[M]$ with Newton polytope P has a minimal model if and only if Fine interior of P is not empty, i.e.,

$$\kappa(Z) \geq 0 \Leftrightarrow F(P) \neq \emptyset.$$

Moreover, a minimal model $\widehat{Z}$ (if exists) can be obtained as Zariski closure of Z in some toric embedding $\mathbb{T} \hookrightarrow \widehat{V}$, i.e. in some projective toric variety $\widehat{V}$.

First algorithm to construct a minimal model (Ishii)

The first algorithm is standard:

- Take a projective closure of Z in the toric variety defined by the normal fan Σ_P;
- Take a regular refinement Σ of the normal Σ_P and obtain a smooth projective model $\overline{Z}$ of Z as Zariski closure in toric variety $V = V(\Sigma)$. The condition $F(P) \neq \emptyset$ imply that the adjoint linear system $K_V + \overline{Z}$ is not empty (due to Khovanskiǐ);
- Apply toric MMP to pair $(V, \overline{Z})$ (due to Miles Reid). After finitely many steps we get a toric pair $(\widehat{V}, \widehat{Z})$ with nef adjoint divisor. Then $\widehat{Z}$ is a minimal model.

Second algorithm to construct a minimal model (Ishii)

The second algorithm is shorter. It is important to use the fact that a minimal model X is always uniquely determined in codimension 1.

Definition

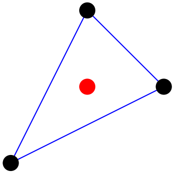
The *support of Fine interior* $F(P)$ is the finite set

$$S_F(P) := \{\nu \in N \ : \ \mathrm{ord}_{F(P)}(\nu) = \mathrm{ord}_P(\nu) + 1\}.$$

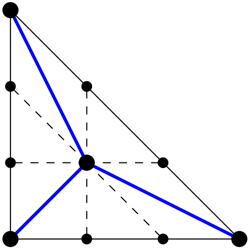
This is the set of *essential valuations* $\nu \in N$ that contribute to $F(P)$

- Find the set $S_F(P)$ as the set of lattice generators of 1-dimensional cones in the fan $\widehat{\Sigma}$.
- Construct the simplicial fan $\widehat{\Sigma}$ with $\widehat{\Sigma}[1] = S_F(P)$ as a normal fan to some full-dimensional simple polytope $\square(\varepsilon)$ with given facet normals $S_F(P)$ obtained by "puffing up" the polytope $F(P)$.

Example in dimension 2



$F(P) \subset P$



$P^*, \quad S_F(P)$

Definition

A d-dimensional lattice polytope $P \subset M_{\mathbb{R}}$ containing $0 \in M$ in its interior is called *reflexive* if the *polar dual* polytope

$$P^* := \{y \in N_{\mathbb{R}} : \langle x, y \rangle \geq -1, \ \forall x \in P\}$$

is a lattice polytope.

Remark

Let P be a reflexive polytope. Then

- $F(P) = \{0\}$;
- $S_F(P) = P^* \cap (N \setminus \{0\})$.

2-dimensional reflexive polytopes

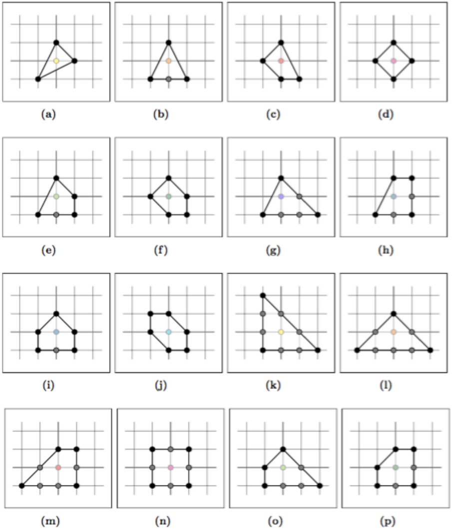
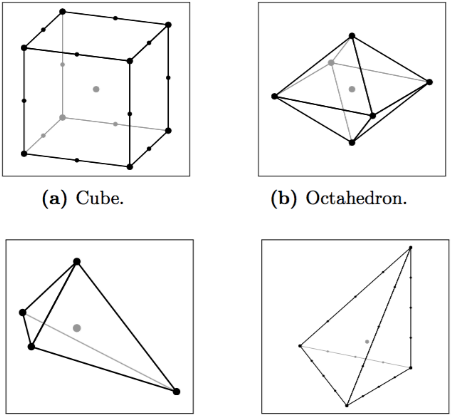


Figure: 2-dimensional reflexive polytopes

Source: Karin Schaller, *Stringy Invariants of Algebraic Varieties and Lattice Polytopes*, Ph.D. thesis, Eberhart Karls Univ. Tübingen, 2019.

Some Reflexive 3-polytopes



(a) Cube. **(b)** Octahedron. **(c)** 3-simplex. **(d)** Tetrahedron.

Figure: Some Reflexive 3-polytopes.

Source: Karin Schaller, *Stringy Invariants of Algebraic Varieties and Lattice Polytopes*, Ph.D. thesis, Eberhart Karls Univ. Tübingen, 2019.

The combinatorial duality

If P is reflexive, then P^* is also reflexive and

$$\boxed{(P^*)^* = P.}$$

There exists a natural 1-to-1 correspondence between k-dimensional faces $Q \prec P$ and $(d-k-1)$-dimensional faces $Q^* \prec P^*$:

$$Q^* := \{y \in P^* \ : \ \langle x, y \rangle = -1 \ \ \forall x \in Q\}.$$

The combinatorial duality $P \leftrightarrow P^*$ perfectly agrees with the prediction of MS for Calabi-Yau hypersurfaces in toric varieties $X \subset V_P$ and $X^* \subset V_{P^*}$.

Minimal models in case of reflexive polytopes (B.1994)

- Take the Zariski closure $\widetilde{Z}$ of Z in the Gorenstein toric Fano variety $\widetilde{V}$ defined by the normal fan of P. We call $\widetilde{Z}$ *canonical model of Z*;
- Take a maximal projective simplicial subdivision $\widehat{\Sigma}$ of Σ_P with $\widehat{\Sigma}[1] = S_F(P)$. The Zariski closure of Z in the toric variety $\widehat{V}$ is a (Calabi-Yau) *minimal model of Z*.

Third algorithm to construct minimal model (B.)

Definition

Let P be a lattice polytope with $F(P) \neq \emptyset$. We define *canonical hull* $C(P)$ of P as

$$C(P) := \{x \in M_{\mathbb{R}} \ : \ \langle x, \nu \rangle \geq \mathrm{ord}_P(\nu) \ \ \forall \nu \in S_F(P)\}.$$

Remark

$C(P)$ is a rational polytope containing P.

One has $P = C(P)$ if $\dim P = 2$, or if P is reflexive, but in general $C(P)$ is larger as P.

Third algorithm to construct minimal model (B.)

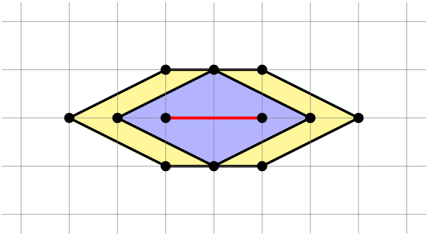
Definition

$$\widetilde{P} := C(P) + F(P).$$

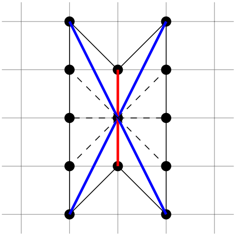
Theorem

The normal fan $\widetilde{\Sigma}$ of $\widetilde{P}$ is a $\mathbb{Q}$-Gorenstein toric variety with at worst canonical singularities. One $\widetilde{\Sigma}[1] \subset S_F(P)$. Any simplicial subdivision $\widehat{\Sigma}$ of $\widetilde{\Sigma}$ with $\widehat{\Sigma}[1] = S_F(P)$ defines a crepant morphism $\widehat{V} \to \widetilde{V}$ of the corresponding toric varieties. The simplicial toric variety $\widehat{V}$ has at worst terminal singularities. The Zariski closure $\widehat{Z}$ of Z in $\widehat{V}$ is a *minimal model* of Z.

Example dim $F(P) = 1$

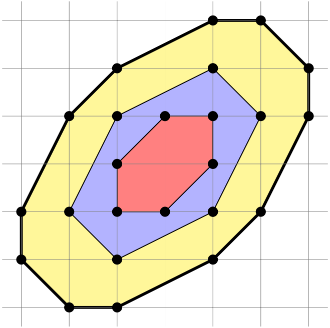


$F(P) \subset P \subset \widetilde{P} = F(P) + P$

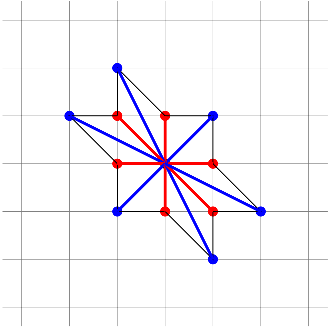


$S_F(P)$, $\widetilde{\Sigma}$

Example $\dim F(P) = 2$

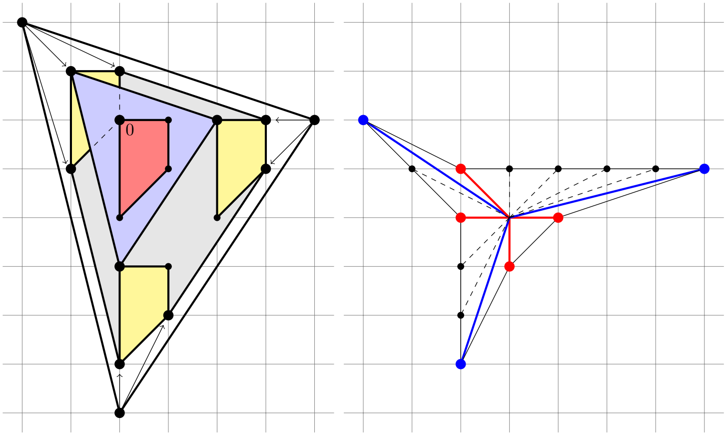


$F(P) \subset P \subset \widetilde{P} = F(P) + P$



$S_F(P)$, $\widetilde{\Sigma}$

Idea of proof



Applications

- There exist a unique *canonical model* $\widetilde{Z}$ of any non-degenerate toric hypersurface Z with the Newton polytope P if $F(P) \neq \emptyset$.
- Kodaira dimension of $\widehat{Z}$ equals $\kappa = \min\{d-1, \dim F(P)\}$.
- The *Iitaka fibration* of $\widetilde{Z} \to V_{F(P)}$ is induced by the natural toric morphism $\widetilde{V} \to V_{F(P)}$ (canonical toric Fano fibration).
- Generic *fibers of the Iitaka fibrations* are $(d-1-\kappa)$-dimensional canonical non-degenerate toric hypersurfaces of Kodaira dimension 0.

Mirror Symmetry

see arXiv:2006.15825 (Today)

Thank you !