Algebraic Number Theory
Example Sheet 2

Hand in the answers to questions 3, 5, 7 (marked with †).
Deadline 2pm Friday, Week 6.

1. Let $\sigma : \mathbb{Q}(\sqrt{5}) \hookrightarrow \mathbb{C}$ be given by $\sigma(a + b\sqrt{5}) = a - b\sqrt{5}$. Explicitly write down the embeddings $\tau : \mathbb{Q}(\sqrt{5}, \sqrt{6}) \hookrightarrow \mathbb{C}$ that extend σ.

2. Which integers $-10 \leq D \leq 10$ are discriminants of quadratic fields?

†3. Let $K = \mathbb{Q}(\alpha)$ where α is a root of $X^3 - 2X - 2$. Compute the trace $\mathrm{Tr}_{K/\mathbb{Q}}$ of $1, \alpha, \alpha^2, \alpha^3, \alpha^4$ and hence calculate $\Delta(1, \alpha, \alpha^2)$. Determine $\mathcal{O}_K$ and Δ_K.

4. Suppose $f = X^3 + bX + c \in \mathbb{Q}[X]$ is irreducible and let α be a root. Let $K = \mathbb{Q}(\alpha)$. Show that
$$\Delta(1, \alpha, \alpha^2) = -4b^3 - 27c^2.$$

†5. Let α be as in Q3. Show carefully that $\mathbb{Q}(\alpha) \neq \mathbb{Q}(\sqrt[3]{d})$ for any non-cube d. Thus not all cubic fields are of the form $\mathbb{Q}(\sqrt[3]{d})$.
Hint: Let $\eta = \sqrt[3]{d}$ and suppose $K = \mathbb{Q}(\eta)$. Then $1, \alpha, \alpha^2$ and $1, \eta, \eta^2$ are both $\mathbb{Q}$-bases for K. What do we know about the ratio $\Delta(1, \alpha, \alpha^2)/\Delta(1, \eta, \eta^2)$?

6. Let $\alpha = \sqrt[3]{10}$. Show that $(1 + \alpha + \alpha^2)/3$ is an algebraic integer. Compute an integral basis for $K = \mathbb{Q}(\alpha)$. What is Δ_K? (The answer is -300).

†7. Let p be an odd prime and let $\zeta = \zeta_p = \exp(2\pi i/p)$. Let $K = \mathbb{Q}(\zeta)$ and let $\omega = \zeta - 1$. You may want to make use of question 4 from example sheet 1 while doing this question, and you may assume that $\mathrm{Nm}_{K/\mathbb{Q}}(\omega) = p$.

   (i) Explain why the conjugates of ζ are
   $$\zeta,\ \zeta^2,\ \zeta^3,\ \ldots,\ \zeta^{p-1}.$$

   (ii) Using the determinant of a Vandermonde matrix, show that
   $$\Delta(1, \zeta, \ldots, \zeta^{p-2}) = \prod_{1 \leq i < j \leq p-1} (\zeta^i - \zeta^j)^2 = (-1)^{(p-1)/2} \cdot \prod_{\substack{1 \leq i,j \leq p-1, \\ i \neq j}} (\zeta^i - \zeta^j).$$

   (iii) Prove that
   $$\Delta(1, \zeta, \ldots, \zeta^{p-2}) = (-1)^{(p-1)/2} \left(\prod_{i=1}^{p-1} \zeta^j \right)^{p-2} \cdot \left(\prod_{k=1}^{p-1} (\zeta^k - 1) \right)^{p-2}.$$
   Express this in terms of $\mathrm{Nm}_{K/\mathbb{Q}}(\zeta)$ and $\mathrm{Nm}_{K/\mathbb{Q}}(\omega)$ and deduce that
   $$\Delta(1, \zeta, \ldots, \zeta^{p-2}) = (-1)^{(p-1)/2} p^{p-2}.$$

   (iv) Using the fact that the minimal polynomial of ω is Eisenstein at p, show that $\frac{\omega^{p-1}}{p}$ is an algebraic integer.

(v) Suppose that

$$\beta = \frac{u_0 + u_1\omega + \cdots + u_{p-2}\omega^{p-2}}{p}$$

is an algebraic integer, with $u_i \in \mathbb{Z}$ not all zero and $0 \le u_i < p$. Let j be the smallest nonnegative integer such that $u_j \neq 0$.

(a) By considering $\omega^{p-2-j}\beta$, show that $\frac{u_j\omega^{p-2}}{p}$ is an algebraic integer.

(b) Calculate $\mathrm{Nm}_{K/\mathbb{Q}}(\frac{u_j\omega^{p-2}}{p})$ and obtain a contradiction.

(vi) Show that $1, \zeta, \ldots, \zeta^{p-2}$ is an integral basis for K.

8. Let K be a number field. We say that K is **totally real** if all its embeddings are real. Show that if K is totally real then the discriminant Δ_K is positive.

9. Let R be a ring and $\mathfrak{a}$ be an ideal of R. Show that $\mathfrak{a} = R$ if and only if $\mathfrak{a}$ contains a unit.

10. Let ω be an algebraic integer.
    (i) Show that some conjugate of ω has absolute value ≥ 1.
    (ii) Suppose further that $\mathrm{Nm}(\omega) = 1$. Show that that some conjugate has absolute value ≤ 1.
    (iii) (Hard!) With the help of (ii), show that $X^n + X + 3$ is irreducible over $\mathbb{Q}$ for all $n \ge 2$.

11. Let, for $n \ge 1$,

$$M_n = (1+\sqrt{2})^n + (1-\sqrt{2})^n.$$

Show (without expanding brackets) that $M_n \in \mathbb{Z}$, and that moreover it is the nearest integer to $(1+\sqrt{2})^n$.