A conjecture on crystalline lifts of residual Galois representations

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April 25, 2013
Let $K$ be a finite unramified extension of $\mathbb{Q}_p$, and let $S_K := \text{Hom}(K, \overline{\mathbb{Q}}_p)$.

Let $\bar{\rho} : G_K \rightarrow \text{GL}_2(\overline{\mathbb{F}}_p)$ be of the form $\begin{pmatrix} \chi_1 & * \\ 0 & \chi_2 \end{pmatrix}$.

Let $\sigma$ be a Serre weight of $\bar{\rho}$.

**Goal:** Provide numerical evidence for a conjecture which gives necessary and sufficient conditions for $\bar{\rho}$ to admit a crystalline lift $\rho$ with labeled Hodge-Tate weights $\text{HT}_\tau(\sigma) = \{m_\tau, m_\tau + n_\tau\}$ for all $\tau \in S_K$. 

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The conjecture

Let $K$ denote a finite unramified extension of $\mathbb{Q}_p$ of degree $f$ with residue field $k$, and let $\chi : G_K \to \overline{\mathbb{F}}_p^\times$ be any character.

Recall that $G_K$ has a decreasing filtration by closed subgroups $G_K^u$ where

- $G_K^{-1} = G_K$,
- $G_K^u = I_K$ for $-1 < u \leq 0$, and
- $\bigcup_{u>0} G_K^u$ is the wild ramification subgroup $P_K$.

We define an increasing filtration on $H^1(G_K, \chi)$ by setting

$$\text{Fil}^s H^1(G_K, \chi) = \bigcap_{u>s-1} \ker(H^1(G_K, \chi) \to H^1(G_K^u, \chi))$$

for $s \in \mathbb{R}$. 
The conjecture

Facts:

1. $\text{Fil}^s H^1(G_K, \chi) = 0$ for $s < 0$.

2. $\text{Fil}^0 H^1(G_K, \chi) = \ker(H^1(G_K, \chi) \to H^1(I_K, \chi))$.

3. $\text{Fil}^s(G_K, \chi) = \text{Fil}^0(G_K, \chi)$ for $0 \leq s \leq 1$.

For any $s \in \mathbb{R}$, we set

$$\text{Fil}^{<s}(H^1(G_K, \chi)) = \bigcup_{t<s} \text{Fil}^t(H^1(G_K, \chi)).$$

Fact:

$$\text{Fil}^{<s}(H^1(G_K, \chi)) = \ker(H^1(G_K, \chi) \to H^1(G_K^{s-1}, \chi)).$$
The conjecture

We define the following subspaces of $H^1(G_K, \chi)$:

\[
H^1_{\text{nr}}(G_K, \chi) = \text{Fil}^0 H^1(G_K, \chi) = \text{Fil}^1 H^1(G_K, \chi);
\]
\[
H^1_{\text{rr}}(G_K, \chi) = \text{Fil}^{< \frac{p}{p-1}} H^1(G_K, \chi);
\]
\[
H^1_{\text{ar}}(G_K, \chi) = \text{Fil}^{<1+ \frac{p}{p-1}} H^1(G_K, \chi).
\]

We call

- $H^1_{\text{nr}}(G_K, \chi)$: unramified subspace of $H^1(G_K, \chi)$;
- $H^1_{\text{rr}}(G_K, \chi)$: rarement ramifié subspace of $H^1(G_K, \chi)$;
- $H^1_{\text{ar}}(G_K, \chi)$: assez ramifié subspace of $H^1(G_K, \chi)$;

We use the same terminology to describe the cohomology classes in these subspaces.

Note that:

\[
H^1_{\text{nr}}(G_K, \chi) \subset H^1_{\text{rr}}(G_K, \chi) \subset H^1_{\text{ar}}(G_K, \chi).
\]
The conjecture

Choose an embedding $\tau_0 : k \to \overline{F}_p$, let $\tau_i = \text{Frob}^i \circ \tau_0$ where Frob is the absolute Frobenius on $\overline{F}_p$.

Let $\omega_0 : I_K \to \overline{F}_p^\times$ denote the fundamental character associated with $\tau_0$. Then, we have

$$\omega_i = \tau_i \circ \omega_0 \text{ for all } i = 0, \ldots, f - 1.$$ 

We may then write

$$\chi|_{I_K} = \prod_{j=0}^{f-1} \omega_i^{n_j} = \omega_0^n, \text{ where } n = \sum_{j=0}^{f-1} n_j p^j,$$

for integers $a_j$ satisfying $1 \leq n_j \leq p$ for $j = 0, \ldots, f - 1$, with some $n_j < p$. We will often view the indices $i$ as being in $\mathbb{Z}/f\mathbb{Z}$. 

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The conjecture

Proposition

With the above notation and \( n_j \neq p \) (resp. \( n_j \neq 1 \)) for some \( j \), we have

1. \( H^1_{\text{nr}}(G_K, \chi) = 0 \) unless \( \chi \) is trivial, in which case \( H^1_{\text{nr}}(G_K, \chi) \) has dimension 1;

2. \( H^1(G_K, \chi) = H^1_{\text{ar}}(G_K, \chi) \) unless \( \chi \) is cyclotomic, in which case \( H^1(G_K, \chi)/H^1_{\text{ar}}(G_K, \chi) \) has dimension 1;

3. \( H^1_{\text{ar}}(G_K, \chi)/H^1_{\text{nr}}(G_K, \chi) \) has dimension \( f \);

4. \( H^1_{\text{rr}}(G_K, \chi)/H^1_{\text{nr}}(G_K, \chi) \) has dimension equal to the number of \( i \in \{0, \ldots, f - 1\} \) such that \( n_i = p \).
The conjecture

Let $\bar{\rho} : G_K \to \GL_2(\overline{\mathbb{F}}_p)$ be of the form $\begin{pmatrix} \chi_1 & * \\ 0 & \chi_2 \end{pmatrix}$, and write $\chi = \chi_1 \chi_2^{-1}$.

Let $\sigma$ be a **Serre weight** for $W(\bar{\rho})$, so that

$$\sigma = \sigma_{\vec{a}, \vec{b}} = \bigotimes_{i=0}^{f-1} (\det a_i \otimes \Sym^{b_i-1} \overline{\mathbb{F}}_p)$$

where the action of $\GL_2(k)$ on the $i$th factor is via $\tau_i = \Frob^i \circ \tau_0$ and $1 \leq b_i \leq p$ for all $i$. Then, we have

$$\chi_1|_{I_K} = \prod_{i=0}^{f-1} \omega_i^{a_i} \prod_{j \in J} \omega_j^{b_j}, \quad \chi_2|_{I_K} = \prod_{i=0}^{f-1} \omega_i^{a_i} \prod_{j \not\in J} \omega_j^{b_j}$$

for some $J \subset \{0, \ldots, f-1\}$. 
The conjecture

Note that the same $(\vec{a}, \vec{b})$ can arise for multiple $J$!

But: Gee-Liu-Savitt proves there is a unique such $J = J_{\text{max}}(\sigma)$ subject to the conditions:

1. if $(b_j, b_{j+1}, \ldots, b_i) = (p, p - 1, \ldots, p - 1, 1)$ with $j, j + 1, \ldots, i - 1 \notin J$, then $i \notin J$;
2. if $(b_0, \ldots, b_{f-1}) = (p - 1, \ldots, p - 1)$ then $J \neq \emptyset$.

Note that Condition (2) is only relevant if $\chi$ is unramified (all $n_i = p - 1$).
The conjecture

Let $W(\bar{\rho})$ be the set of Serre weights of $\bar{\rho}$.

**Consequence.** The map

$$W(\bar{\rho}) \rightarrow \text{Subsets}(\{0, \ldots, f - 1\})$$

$$\sigma \mapsto J_{\text{max}}(\sigma)$$

is injective, except in the case where $\chi|_{I_K} = \chi_{\text{cycl}}$ (resp. $\chi|_{I_K}^{-1} = \chi_{\text{cycl}}$).

Exceptional cases,

- $\chi|_{I_K} = \chi_{\text{cycl}}$ (resp. $\chi|_{I_K}^{-1} = \chi_{\text{cycl}}$), i.e. all $a_i = 1$ (resp. $a_i = p - 2$), where the $\sigma$ with both $n_i = 1$ and $n_i = p$ both arise. In that case, $J_{\text{max}}(\sigma) = \{0, \ldots, f - 1\}$ (resp. $J_{\text{max}} = \emptyset$).
The conjecture

Goal: Describe the subspace \( L_{\sigma} \subset H^1(G_K, \chi) \) corresponding to the classes \( c_{\bar{\rho}} \) such that \( \bar{\rho} \) has a crystalline lift with labeled Hodge-Tate weights \((m_i, m_i + n_i)\) for \( i = 0, \ldots, f - 1 \).

Case 1. All \( n_i = p \) and \( J_{\text{max}}(\sigma) = \{0, \ldots, f - 1\} \).

In that case, \( L_{\sigma} = H^1(G_K, \chi) \), so we can ignore this case.
Recall that we have the inclusions:

\[ H_{nr}^1(G_K, \chi) \subset H_{rr}^1(G_K, \chi) \subset H_{ar}^1(G_K, \chi). \]

Also note that:
- \( L_{\sigma} \) is always contained in \( H_{ar}^1(G_K, \chi) \) (which is all of \( H^1(G_K, \chi) \) anyway unless \( \chi \) is cyclotomic).
- \( L_{\sigma} \) always contains the \( H_{nr}^1(G_K, \chi) \) (which is 0 unless \( \chi \) is ramified).
- By Proposition 1, the image of \( L_{\sigma} \) in \( H_{ar}^1/H_{nr}^1 \) always has dimension \(|J|\) where \( J = J_{\text{max}}(\sigma) \).
- We will define \( L_{\sigma} \) as an intersection, where all the subspaces involved contain \( H_{rr}^1/H_{nr}^1 \).

So, it is enough to define \( L_{\sigma} \) as a subspace of \( H_{ar}/H_{rr} \).
Let $M$ be any extension of $K$ of degree prime to $p$ and ramification degree $e$ dividing $p^f - 1$ such that $\chi$ is trivial on $G_M$.

Suppose also that $\pi$ is a uniformizer of $M$ such that $\pi^e \in K^\times$.

Let $L$ be the maximal unramified subextension of $M$ and let $k_L$ be its residue field. Then:

\[
\begin{align*}
H^1(G_K, \chi) &= \text{Hom}_{\mathbb{F}_p}[\text{Gal}(M/K)](M^\times/(M^\times)^p, \overline{F}_p(\chi)) \\
H^1/H^1_{nr} &= \text{Hom}_{\mathbb{F}_p}[\text{Gal}(M/K)](O_M^\times/(O_M^\times)^p, \overline{F}_p(\chi)) \\
H^1_{ar}/H^1_{rr} &= \text{Hom}_{\mathbb{F}_p}[\text{Gal}(M/K)](U_t/U_{e+t}((O_M^\times)^p \cap U_t), \overline{F}_p(\chi))
\end{align*}
\]

where $t = \lceil e/(p - 1) \rceil$ and $U_m = 1 + \pi^m O_M$. 
Recall that the Artin-Hasse exponential $E_p$ induces an isomorphism
\[ \pi^t \mathcal{O}_M / \pi^{t+e} \mathcal{O}_M \cong U_t / U_{t+e}, \]
given on this quotient by the formula
\[ E_p(x) = \sum_{i=0}^{p-1} \frac{x^i}{i!}. \]
So, the idea is to use $E_p$ to provide a splitting of the relevant part of the ramification filtration.
The conjecture

Let \( i \notin J \), so that \( b_{i+1} \neq p \); and put

\[
    r_i = \sum_{j=0}^{f-1} b_{i+j+1} p^j. 
\]

- Assumption on \( b_{i+1} \) implies that \( p \nmid r_i \).
- Assumption that \( \chi \) is trivial on \( G_M \) implies that \( r_i e/(p^f - 1) \in \mathbb{Z} \).
- Moreover \( r_i e/(p^f - 1) \in [t, t+e) \).

So, the restriction of \( E_p \) to \( k_L \pi^{r_i} \) composed with the projection to \( U_t/U_{e+t}((\mathcal{O}_M^\times)^p \cap U_t) \) is injective.

Hence, we get a surjective map

\[
    H^1_{\text{ar}}/H^1_{\text{rr}} \rightarrow \text{Hom}_{\mathbb{F}_p}[\text{Gal}(M/K)](k_L \pi^{r_i}, \overline{F}_p(\chi)).
\]
The conjecture

We also have a natural isomorphism $k_L \pi^r_i \cong k_L(\omega^r_i)$ where $\omega$ is the fundamental character $G_K \to k^\times$ associated to the uniformizer $\pi$.

So the above target is

$$\text{Hom}_{\overline{\mathbb{F}}_p[\text{Gal}(M/K)]}(k_L(\omega^r_i) \otimes \overline{\mathbb{F}}_p, \overline{\mathbb{F}}_p(\chi)).$$

Decomposing

$$k_L \otimes \overline{\mathbb{F}}_p \cong \bigoplus_{i=0}^{f-1} (k_L \otimes_{k, \tau_i} \overline{\mathbb{F}}_p),$$

we obtain

$$k_L(\omega^r_i) \otimes \overline{\mathbb{F}}_p \cong \bigoplus_{j=0}^{f-1} (k_L \otimes_{k, \tau_i} \overline{\mathbb{F}}_p)(\omega^r_j).$$

Restricting to the $i+1$ component thus gives a surjection

$$H^1_{\text{ar}}/H^1_{\text{rr}} \to \text{Hom}_{\overline{\mathbb{F}}_p[\text{Gal}(M/K)]}((k_L \otimes_{k, \tau_i+1} \overline{\mathbb{F}}_p)(\omega^r_{i+1}), \overline{\mathbb{F}}_p(\chi)).$$
The conjecture

**Note:** Last space is one-dimensional since

\[ k_L \otimes_{k, \tau} \overline{F}_p \cong \text{Ind}_{\text{Gal}(M/L)}^{\text{Gal}(M/K)} \overline{F}_p, \]

and \( \chi = \omega_{i+1}^r \mu \) for some character \( \mu \) of \( \text{Gal}(L/K) \).

Choosing a basis gives, we get a map

\[ \nu_i : H^1_{\text{ar}}/H^1_{\text{rr}} \to \overline{F}_p, \]

well-defined up to a scalar in \( \overline{F}_p^\times \).

One can check that (again up to a scalar), \( \nu_i \) is independent of the choice of \( M \) and \( \pi \) in its definition.
The conjecture

Conjecture (Diamond)

Let $K$ be a finite unramified extension of $\mathbb{Q}_p$. Let $\bar{\rho} : G_K \to \text{GL}_2(\overline{\mathbb{F}}_p)$ be of the form

$$\begin{pmatrix} \chi_1 & * \\ 0 & \chi_2 \end{pmatrix},$$

and write

$$\chi = \chi_1 \chi_2^{-1}.$$  Let $\sigma \in W(\bar{\rho})$ be a Serre weight. Then $\bar{\rho}$ has a crystalline lift of labeled Hodge-Tate weight $\text{HT}_i(\sigma)$ if and only if the associated extension class $c_{\bar{\rho}} \in H^1(G_K, \chi)$ belongs to

$$L_{\sigma} = \bigcap_{i \notin J_{\text{max}}(\sigma)} \ker \nu_i.$$
First example

This example comes from a global representation over the totally real cubic field $F = \mathbb{Q}(\zeta_7)^+$ in which 3 is inert.

Consider the form in the table below, and let

$$\bar{\rho} : \text{Gal}(\overline{\mathbb{Q}}/F) \to \text{GL}_2(\mathbb{F}_{27})$$

be the associated Galois representation.

The projective image of $\bar{\rho}$ is $\text{PGL}_2(\mathbb{F}_{27})$. So, this cuts out a $\text{PGL}_2(\mathbb{F}_{27}) \rtimes \mathbb{Z}/3\mathbb{Z}$-extension of $\mathbb{Q}$ ramified at 2, 3 and 7 only.

The corresponding Galois representation can be realised in the 3-torsion of some Shimura curve. (It is also given by an explicit polynomial.)
First example

Hecke eigenvalues

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<th>$Np$</th>
<th>13</th>
<th>29</th>
<th>41</th>
<th>43</th>
<th>71</th>
<th>83</th>
<th>97</th>
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<td>$\text{Tr}(P\bar{\rho}(Frob_p))$</td>
<td>$\alpha^{20}$</td>
<td>$\alpha^{9}$</td>
<td>$\alpha^{6}$</td>
<td>$\alpha^{16}$</td>
<td>$\alpha^{5}$</td>
<td>2</td>
<td>$\alpha^{14}$</td>
</tr>
<tr>
<td>$\text{ord}(P\bar{\rho}(Frob_p))$</td>
<td>13</td>
<td>28</td>
<td>26</td>
<td>13</td>
<td>26</td>
<td>4</td>
<td>13</td>
</tr>
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</table>

Serre weights

<table>
<thead>
<tr>
<th>$\vec{a}$</th>
<th>$\vec{b}$</th>
<th>mult.</th>
</tr>
</thead>
<tbody>
<tr>
<td>[0, 1, 1]</td>
<td>[3, 1, 1]</td>
<td>1</td>
</tr>
<tr>
<td>[0, 1, 2]</td>
<td>[2, 1, 2]</td>
<td>1</td>
</tr>
<tr>
<td>[0, 2, 2]</td>
<td>[2, 2, 1]</td>
<td>1</td>
</tr>
<tr>
<td>[0, 2, 0]</td>
<td>[3, 2, 2]</td>
<td>1</td>
</tr>
</tbody>
</table>
Here, we have
\[ \chi|_{I_K} = \omega_0^{17} \text{ so } (b_0, b_1, b_2) = (2, 2, 1). \]

Note that \( b_2 = 1 \), so this is \textbf{not} generic, but \( b_i < p \) for all \( i \).

**So Conjecture 2 gives a a complete description of the spaces \( L_\sigma \) for all possible weights \( \sigma \).**

We can identify \( H^1(G_K, \chi) \) with the \( G \)-equivariant homomorphisms \( M^*/(M^*)^3 \to \overline{\mathbb{F}}_p(\chi) \) where \( K = \mathbb{Q}_{27}, M = K(\pi) \) for some uniformizer \( \pi \) such that \( \pi^{26} \in K \) and \( G = \text{Gal}(M/K) \).
Let $N$ be the splitting field of $P\rho|_{D_3}$.

Then $N = M[\beta]$ where $\beta$ is a root of $x^{27} + 6x^{23} + 6$.

In this case we can decompose the $\chi$-isotypic component of $M^*/(M^*)^3 \otimes \overline{F}_p$ into the direct sum of the three $G$-submodules generated by the images of $E_3(\pi^{17})$, $E_3(\pi^{23})$ and $E_3(\pi^{25})$.

Let $c \in H^1(G, \chi)$; then chasing through the definitions, one finds that:

$$c \in \ker(\nu_0) \text{ (resp. } \ker(\nu_1), \ker(\nu_2)) \iff E_3(\pi^{23}) \text{ (resp. } E_3(\pi^{25}), E_3(\pi^{17})) \text{ is in } \ker \phi_c,$$

where $\phi_c$ is the homomorphism corresponding to $c$ (after trivialisation).

This is equivalent to the corresponding unit being in the image of $\text{Norm} : N^* \to M^*$. 
First example

In our example, explicit computations show that:

- $E_3(\pi^{25})$ and $E_3(\pi^{17})$ are norms, so $c_{\bar{\rho}} \in \ker(\nu_1) \cap \ker(\nu_2)$.
- But $E_3(\pi^{23})$ is not a norm, so $c_{\bar{\rho}} \notin \ker(\nu_0)$.

Now here is the list of possible weights $\sigma$ (up to twist), with the corresponding $J_{\text{max}}(\sigma)$:

- $(2,2,1), \{0,1,2\}$
- $(3,2,2), \{0,1\}$
- $(2,1,2), \{0,2\}$
- $(1,3,1), \{1,2\}$
- $(3,1,1), \{0\}$
- $(1,3,3), \{2\}$
- $(2,2,3), \emptyset$

(there’s no weight with $J_{\text{max}} = 1$)
From the norm calculation discussed above, we expect to get precisely the weights $\sigma$ with $0 \in J_{\text{max}}(\sigma)$, and indeed this matches the list of 4 weights that appear in the Hecke data.

**Note:** We also have many other examples with $p = 2!$