Theta lifts of automorphic forms and applications to paramodularity

L. Dembélé

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September 9, 2014
**Modularity of elliptic curves**

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Motivation

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Eichler-Shimura construction
# Motivation

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Eichler-Shimura construction

Modularity: Shimura-Taniyama-Weil, Wiles et al.
Motivation

There are (at least) two ways in which one could generalise this picture:

1. A non-CM elliptic curve $E/\mathbb{Q}$ is an abelian variety of $\text{GL}_2$-type, i.e. $\text{End}_\mathbb{Q}(E) = \mathbb{Z}$ is an order in $\mathbb{Q}$! This approach leads to the so-called Eichler-Shimura and Shimura-Taniyama-Weil type conjectures.
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2. A non-CM elliptic curve $E/\mathbb{Q}$ is a curve of genus one such that $\text{Jac}(E) = E$ and $\text{End}_{\mathbb{Q}}(E) = \mathbb{Z}$ is trivial! This approach leads to the so-called Paramodularity Conjecture.
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The first attempt to generalise this to abelian surfaces is due to Yoshida (81). But this was quickly proved to be wrong: The level structure was incorrect!
Motivation

Paramodularity conjecture

Conjecture (Brumer-Kramer)

Let $g$ be a paramodular Siegel newform of genus 2, weight 2 and level $N$, with integer Hecke eigenvalues, which is not in the span of Gritsenko lifts. Then there exists an abelian surface $B$ defined over $\mathbb{Q}$ of conductor $N$ such that $\text{End}_\mathbb{Q}(B) = \mathbb{Z}$ and $L(g, s) = L(B, s)$.

Conversely, let $B$ be an abelian surface defined over $\mathbb{Q}$ with $\text{End}_\mathbb{Q}(B) = \mathbb{Z}$. Then there exists an integer $N > 1$, a Siegel newform $g$ of genus 2, weight 2 and paramodular level $N$ such that $L(g, s) = L(B, s)$. 
**Paramodularity conjecture**

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Motivation

Paramodularity conjecture

Let $B$ be an abelian surface defined over $\mathbb{Q}$ of conductor $N$ such that $\text{End}_{\mathbb{Q}}(B) = \mathbb{Z}$, and $p \geq 5$ a prime.

- Under some technical conditions, Tilouine shows that the Galois representation on the $p$-adic Tate module of $A$ comes from an overconvergent Siegel cusp form $g$ of weight 2.
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- Under some technical conditions, Tilouine shows that the Galois representation on the \( p \)-adic Tate module of \( A \) comes from an \textit{overconvergent} Siegel cusp form \( g \) of weight 2.

- It remains to show that \( g \) is \textit{classical}! In that direction, there is work of Tilouine, Pilloni et al.
Brumer and Kramer: Conjecture 1 should be verifiable for abelian surfaces $B$ over $\mathbb{Q}$ with $\text{End}_{\mathbb{Q}}(B) \supseteq \text{End}_{\mathbb{Q}}(B) = \mathbb{Z}$.

From now on, assume that:

1. $F$ is a quadratic field;
2. $A = B \otimes_{\mathbb{Q}} F$ is of $\text{GL}_2$-type over $F$: $\text{End}_F(A)$ is an order in a quadratic field.

**Question:** What can we say about Brumer-Kramer’s observation?
Conjecture (**GL**₂-modularity)**

Let $A$ be an abelian surface of **GL**₂-type. Then, there exists a cuspidal automorphic form $f$ of level $\mathfrak{N}$ and weight 2 on **GL**₂($\mathbb{A}_F$) such that

$$L(A, s) = L(f, s)L(f^\tau, s),$$

where $K$ is the quadratic field generated by the coefficients of $f$ and $\text{Gal}(K/\mathbb{Q}) = \langle \tau \rangle$.

The form $f$ is a:

- **Hilbert cusp form** when $F$ is real.
- **Bianchi cusp form** when $F$ is imaginary.
So when $B$ is an abelian surface over $\mathbb{Q}$ such that

1. $\text{End}_\mathbb{Q}(B) = \mathbb{Z}$, and
2. $\text{End}_F(A)$ is an order in a quadratic field,

one expects that Conjectures 1 and 2, and hence the forms $f$ and $g$, to be related.

Indeed, fix a prime $p$ and consider the Galois representations

$$
\rho_{A,p} : \text{Gal}(\overline{\mathbb{Q}}/F) \to \text{GL}_2(\overline{\mathbb{Q}}_p)
$$

$$
\rho_{B,p} : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \text{GSp}_4(\overline{\mathbb{Q}}_p).
$$

Then, we see that $\rho_{B,p} = \text{Ind}^G_{G_F} \rho_{A,p}$.

So when $\rho_{A,p}$ is automorphic, we expect $g$ to be a lift of $f$!
Theorem (Johnson-Leung-Roberts)

Let $F / \mathbb{Q}$ be a real quadratic field of discriminant $D$. Let $\mathcal{O}_F$ be the ring of integers of $F$, and $\mathfrak{N} \subseteq \mathcal{O}_F$ an ideal. Let $f$ be a Hilbert newform of level $\mathfrak{N}$, weight $(2, 2k - 2)$ and trivial central character where $k \geq 2$. Then, there exists a Siegel newform $F$ of weight $(k, k)$ and paramodular level $N = D^2 \mathcal{N}_{F/\mathbb{Q}}(\mathfrak{N})$ with Hecke eigenvalues, epsilon factor and (spinor) $L$-function determined explicitly by those of $f$. 
Theorem (Berger-D-Pacetti-Şengün)

Let $F/\mathbb{Q}$ be an imaginary quadratic field of discriminant $D$. Let $\mathcal{O}_F$ be the ring of integers of $F$, and $\mathfrak{N} \subseteq \mathcal{O}_F$ an ideal. Let $f$ be a Bianchi newform of level $\mathfrak{N}$, (even) weight $k \geq 2$ and trivial central character. Then, there exists a Siegel newform $F$ of weight $(k,2)$ and paramodular level $N = D^2 N_{F/\mathbb{Q}}(\mathfrak{N})$ with Hecke eigenvalues, epsilon factor and (spinor) $L$-function determined explicitly by those of $f$. 
Corollary (D.-Kumar)

Assume that Conjecture 2 is true. Let $B$ be an abelian surface defined over $\mathbb{Q}$ such that $\text{End}_{\mathbb{Q}}(B) = \mathbb{Z}$, and the base change of $B$ to some quadratic field is of $\text{GL}_2$-type. Then, $B$ is paramodular.
Proposition (D.-Kumar)

Let $B$ be the Jacobian of the curve

\[ C : \ y^2 = -8x^6 + 220x^5 - 44x^4 - 14828x^3 - 4661x^2 - 21016x + 10028. \]

Then $B$ is a paramodular surface of conductor $193^2$.

Proof.

The proof of this combines Johnson-Leung-Roberts’ result with Conjecture 2.
Proposition (Berger-D.-Pacetti-Şengün)

Let $B$ the Jacobian of the curve $C$ given by

$$C : y^2 = 31x^6 + 952x^5 - 5764x^4 - 3750x^3 + 5272x^2 - 7060x + 4783,$$

Then, $B$ is a paramodular abelian surface of conductor $223^2$. 

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Let $F = \mathbb{Q}(\sqrt{-223})$ and $w = \frac{1+\sqrt{-223}}{2}$, and consider the curve

$$C' : y^2 + Q(x)y = P(x),$$

where

$$P := -8x^6 + (54w - 27)x^5 + 9103x^4 + (-14200w + 7100)x^3 - 697185x^2$$
$$+ (326468w - 163234)x + 3539399,$$

$$Q := x^3 + (2w - 1)x^2 - x.$$

The curve $C'$ is a global minimal model for the base change of $C$ to $F$ and it has everywhere good reduction.

We have $A = \text{Jac}(C').$
The modularity of $B$ is deduced from Theorem 4 and the following result.

**Theorem (Berger-D.-Pacetti-Şengün)**

*The surface $A$ has real multiplication by $\mathbb{Z}[\sqrt{2}]$, and there exists a Bianchi newform $f$ of level $(1)$ and weight $2$ and trivial central character such that $f^\sigma = f^\tau$ and*

$$L(A, s) = L(f, s)L(f^\tau, s),$$

*where $\langle \sigma \rangle = \text{Gal}(F/\mathbb{Q})$ and $\langle \tau \rangle = \text{Gal}(\mathbb{Q}(\sqrt{2})/\mathbb{Q})$."

**Proof.**

To prove Theorem 8, we use the so-called Faltings-Serre methods. This is due to the lack of modularity results for imaginary quadratic fields.
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<td>$-8x^6 + 172x^5 + 1760x^4 + 6296x^3 - 44531x^2 - 19128x + 134836$</td>
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<td>$277^2$</td>
<td>$-96x^6 + 2092x^5 - 11820x^4 + 516x^3 + 36076x^2 + 3916x + 14297$</td>
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<tr>
<td>$349^2$</td>
<td>$-1336x^6 - 336x^5 - 7592x^4 - 11244x^3 - 9998x^2 - 37361x - 21356$</td>
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