Blow up of solutions for a 1D transport equation with nonlocal velocity and supercritical dissipation

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Abstract

We study a 1D transport equation with nonlocal velocity and supercritical dissipation. We show that for a certain class of initial data the solution blows up in finite time.

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1991 MSC: 35Q35

1 Introduction and main results

In this paper we consider the following one dimensional equation

$$\theta_t + (H\theta)\theta_x = -\kappa\Lambda^\gamma\theta, x \in \mathbb{R}, \tag{1}$$

where $H\theta$ is the Hilbert transform defined by

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$$H\theta := \frac{1}{\pi} P.V. \int_{-\infty}^{\infty} \frac{\theta(y)}{x-y} dy,$$

κ is a positive number and $\Lambda^\gamma \theta = (-\Delta)^{\gamma/2}\theta$. We consider the equation with a plus sign for the nonlinear term rather than the a priori more physically meaninful with a minus to simplify our presentation. It is clear that due to the properties of the Hilbert transform the transformation $\theta \to -\theta$ transforms one equation into the other.

The model equation (1) corresponds to a simple 1D version of the usual surface quasi-geostrophic equation and by extension of 3D Euler. We refer the reader to (9) and references therein for recent results on the 2D quasi-geostrophic equation. (1) Also has some strong connections with the Birkhoff-Rott equation (see (6) for more details). Other models that have been proposed to understand 3D Euler are

$$\omega_t = \omega H \omega$$

proposed by Constantin, Lax and Majda (CLM85) and more directly related to (1)

$$\omega_t + v\omega_x = \omega H \omega$$

$$v = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\int_{-\infty}^{x} w(y,t)dy}{x-z} dz$$

proposed by de Gregorio ((7) and (8)). For these and some other 1D models all of which have some analogy with the 2D quasi-geostrophic equation and the 3D Euler equation in vorticity form, we refer the readers to (2), (3), (5), (8), (10), (11) and (12). As remarked in (6) equation (1) represents the simplest model of a transport equation with nonlocal velocity and a viscous term with powers of the Laplacian.

For $\kappa > 0$, the cases $\gamma < 1$, $\gamma = 1$, $\gamma > 1$ are called supercritical, critical and subcritical respectively. In the inviscid case ($\kappa = 0$), Córdoba, Córdoba and Fontelos (6) proved the existence of blow-up solutions to (1) for a generic class of initial data. They also obtained the global well posedness in the subcritical case. For the critical case, global well-posedness can be proved by adapting the method of (9) and usual estimates. For the supercritical case the local well-posedness in the critical sobolev space $H^{3/2-\gamma}$ and some smoothing estimates are proved in (4). However the following question remained open:

Problem 1 *For the supercritical case $0 \leq \gamma < 1$, does solution of* (1) *with smooth initial data blow up in finite time?*

We shall show that this is indeed the case for $0 \leq \gamma < 1/2$. Our proof will follow the approach of Córdoba, Córdoba and Fontelos, (6) with some modifications needed to avoid the problems introduced by the presence of the diffusion term. We will take the initial data θ_0 to be a positive, smooth, even and compactly supported function on $\mathbb{R}$ with $\theta_0(0) = 0$ and $\|\theta\|_{L^\infty} = M$, where M is an arbitrary positive number. Our result is

Theorem 2 (Main theorem) *Let $0 \leq \gamma < 1/2$. Let $0 < \delta < 1 - 2\gamma$ be arbitrary but fixed. Then there exists a constant $C = C(\gamma, \delta, M) > 0$ such that If the initial data θ_0 satisfies the above conditions with*

$$\int_0^\infty \frac{\theta_0(x)}{x^{1+\delta}} dx > C,$$

then the solution to (1) *blows up in finite time.*

As in (6), our main idea is to consider the evolution of an integral of the solution multiplied by a properly chosen singular weight. The integral satisfies some ordinary differential equation, and it blows up in finite time under the above mentioned conditions on the initial data. From the blow up of the integral we easily conclude the blowup of the first spatial derivative of the solution, due to the Beal-Kato-Majda criterion (1) (4). We remark that our assumptions on θ_0 could be considerably relaxed. As far as we know, this is the first blow-up result on the supercritical models of the quasi-geostrophic equation.

2 Proof of the main theorem

We will argue by contradiction. Under the assumption of the global existence of smooth solutions we will conclude that $\|\theta_x\|_{L^\infty}$ is unbounded as a function of time. The proof of the local wellposedness in the critical space $H^{3/2-\gamma}$ and a Beal-Kato-Majda criterion can be found in (4). We state it here as a lemma.

Lemma 3 *Assume $\theta_0 \in H^{3/2-\gamma}$. Then there exists $T > 0$ such that equation* (1) *has a unique solution in $C([0,T), H^{3/2-\gamma}(\mathbb{R})) \cap C((0,T); H^m(\mathbb{R}))$, $\forall m \geq 3/2 - \gamma$. Furthermore if T^* is the first time the solution cannot be continued in $C((0,T^*); H^m)$, then necessarily we have*

$$\int_0^{T^*} \|\theta_x(t, \cdot\|_{L^\infty} dt = \infty.$$

By lemma 3 it is enough for us to assume that θ_0 is a smooth function. Since θ_0 is an even function on $\mathbb{R}$, it is obvious by the properties of the Hilbert transform that $\theta(t,\cdot)$ is also an even function. This will be important for later constructions. As a first step, we will prove that

$$\int_0^\infty \frac{\theta(x,t)-\theta(0,t)}{x^{1+\delta}}dx$$

becomes infinite in finite time, for some positive δ, depending on γ. We divide (1) by $x^{1+\delta}$. We have

$$\partial_t \int_0^\infty \frac{\theta(x,t)-\theta(0,t)}{x^{1+\delta}}dx = -\int_0^\infty \frac{(H\theta)\theta_x}{x^{1+\delta}}dx - \kappa \int_0^\infty \frac{(\Lambda^\gamma\theta)(t,x)-(\Lambda^\gamma\theta)(t,0)}{x^{1+\delta}}dx$$

In order to estimate the nonlinear term in (1) we will use the following lemma whose proof is a version of Lemma 2.2 found in (6).

Lemma 4 *Let f be an even function on $\mathbb{R}$. Then for any $0<\delta<1$, there exists a constant C_δ such that*

$$-\int_0^\infty \frac{f'(x)(Hf)(x)}{x^{1+\delta}}dx \geq C_\delta \int_0^\infty \frac{1}{x^{2+\delta}}(f(x)-f(0))^2 dx.$$

In order to handle the difussion term, we will use the following lemma

Lemma 5 *For any $\epsilon>0$, $0\leq\gamma<\frac{1}{2}$, $0<\delta<1-2\gamma$, there exist constants C_1, $C_2>0$ depending only on (ϵ,δ,γ), such that*

$$\int_0^\infty \frac{f(x)}{x^{1+\delta+\gamma}}dx \leq \epsilon \int_0^\infty \frac{f(x)^2}{x^{2+\delta}}dx + C_1(1+\|f\|_{L^\infty}),$$

and

$$\int_0^\infty \frac{g(x)^2}{x^{2+\delta}}dx \geq C_2\left(\int_0^\infty \frac{g(x)}{x^{1+\delta}}dx\right)^2 - C_1(1+\|g\|_{L^\infty}^2),$$

for any $f,g\in C^1(R^+)\cap L^\infty(R^+)$ with $f(0)=g(0)=0$.

PROOF.

We shall only prove the first inequality. The second one can be proved similarly. To this end, denote by letter C the constants which may vary from line to line but only depend on $(\epsilon,\ \delta,\ \gamma)$. Then we have

$$\begin{aligned}\int_0^\infty \frac{u(x)}{x^{1+\delta+\gamma}}dx &\leq \int_0^1 \frac{|u(x)|}{x^{1+\frac{\delta}{2}}}x^{-\frac{\delta}{2}-\gamma}dx + C\|u\|_\infty \\ &\leq \epsilon \int_0^1 \frac{u(x)^2}{x^{2+\delta}}dx + C(1+\|u\|_{L^\infty}).\end{aligned}$$

The lemma is proved. □

With the above lemmas we are now ready to prove our main theorem.

PROOF. [proof of the main theorem]

Fix $0 < \delta < 1-2\gamma$ and define

$$J(t) := \int_0^\infty \frac{\theta(t,x)-\theta(t,0)}{x^{1+\delta}}dx.$$

By lemma 4 we compute

$$\begin{aligned}\frac{dJ(t)}{dt} &= -\int_0^\infty \frac{(H\theta)\theta_x}{x^{1+\delta}}dx - \kappa\int_0^\infty \frac{(\Lambda^\gamma\theta)(t,x)-(\Lambda^\gamma\theta)(t,0)}{x^{1+\delta}}dx \\ &\geq C_\delta\int_0^\infty \frac{(\theta(t,x)-\theta(t,0))^2}{x^{2+\delta}}dx - \kappa\int_0^\infty \frac{(\Lambda^\gamma\theta)(t,x)-(\Lambda^\gamma\theta)(t,0)}{x^{1+\delta}}dx \\ &= C_\delta\int_0^\infty \frac{(\theta(t,x)-\theta(t,0))^2}{x^{2+\delta}}dx - \kappa\int_0^\infty \frac{\theta(t,x)-\theta(t,0)}{x^{1+\delta+\gamma}}dx.\end{aligned}$$

The last equality follows from the obvious identity for fractional derivatives: $\Lambda^\delta(\Lambda^\gamma f)|_{x=0} = \Lambda^{\delta+\gamma}f|_{x=0}$. Now apply lemma 5 to $u(x) = \theta(t,x)-\theta(t,0)$ and choose $\epsilon = C_\delta/2$, we have for some constants $C_3 = C_3(\delta,\gamma) > 0$, $C_4 = C_4(\delta,\gamma) > 0$,

$$\frac{dJ(t)}{dt} \geq C_3 J(t)^2 - C_4(1+\|\theta_0\|_{L^\infty}^2), \tag{2}$$

where we have also used the fact that $\|\theta(t,\cdot)\|_{L^\infty} \leq \|\theta_0\|_{L^\infty}$ for any $t \geq 0$. Now it is obvious that if we choose $\theta_0 \in C_c^\infty(R^+)$ with $\|\theta_0\|_{L^\infty} = 1$, $\theta_0(0) = 0$, and

$$J(0) = \int_0^\infty \frac{\theta_0(x)}{x^{1+\delta}} dx$$

to be sufficiently large, then $J(t)$ in (2) will blow up in some finite $T < \infty$, i.e., $J(t) \to \infty$ as $t \uparrow T$. Now since

$$\begin{aligned} J(t) &\leq \sup_{0<x<1} \frac{|\theta(t,x) - \theta(t,0)|}{x} \cdot \frac{1}{1-\delta} + \int_1^\infty \frac{2}{x^{1+\delta}} dx \\ &\leq \frac{\|\theta_x(t,\cdot)\|_{L^\infty}}{1-\delta} + \frac{2}{\delta} \end{aligned}$$

we conclude that $\|\theta_x\|_{L^\infty}$ also blows up in finite time, obtaining the desired contradiction and concluding the proof of the theorem.

□

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