Multicolor Ramsey numbers for triple systems

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Abstract

Given an r-uniform hypergraph H, the multicolor Ramsey number $r_k(H)$ is the minimum n such that every k-coloring of the edges of the complete r-uniform hypergraph K_n^r yields a monochromatic copy of H. We investigate $r_k(H)$ when k grows and H is fixed. For nontrivial 3-uniform hypergraphs H, the function $r_k(H)$ ranges from $\sqrt{6k}(1+o(1))$ to double exponential in k.

We observe that $r_k(H)$ is polynomial in k when H is r-partite and at least single-exponential in k otherwise. Erdős, Hajnal and Rado gave bounds for large cliques K_s^r with $s \geq s_0(r)$, showing its correct exponential tower growth. We give a proof for cliques of all sizes, $s > r$, using a slight modification of the celebrated stepping-up lemma of Erdős and Hajnal.

For 3-uniform hypergraphs, we give an infinite family with sub-double-exponential upper bound and show connections between graph and hypergraph Ramsey numbers. Specifically, we prove that

$$r_k(K_3) \leq r_{4k}(K_4^3 - e) \leq r_{4k}(K_3) + 1,$$

where $K_4^3 - e$ is obtained from K_4^3 by deleting an edge.

We provide some other bounds, including single-exponential bounds for $F_5 = \{abe, abd, cde\}$ as well as asymptotic or exact values of $r_k(H)$ when H is the bow $\{abc, ade\}$, kite $\{abc, abd\}$, tight path $\{abc, bcd, cde\}$ or the windmill $\{abc, bde, cef, bce\}$. We also determine many new "small" Ramsey numbers and show their relations to designs. For example, the lower bound for $r_6(kite) = 8$ is demonstrated by decomposing the triples of a seven element set into six partial STS (two of them are Fano planes).

1 Introduction, results

An r-uniform hypergraph H is a pair (V, E) where V is a vertex set and $E \subseteq \binom{V}{r}$ is the set of edges. Let K_n^r be the complete r-uniform hypergraph containing all r-subsets of vertices as edges.

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For an edge $\{v_1, v_2, \ldots, v_r\}$ we often write $v_1v_2\ldots v_r$. When $r = 2$, denote K_n^r by K_n. We shall also use the notation $\binom{[n]}{r}$ or $\binom{V}{r}$ for the edge set of K_n^r. An r-uniform hypergraph H is ℓ-partite if its vertex set can be partitioned into ℓ parts (called partite sets) such that each edge contains at most one vertex from each part; H is a complete r-partite hypergraph if each choice of r vertices from distinct partite sets forms an edge, and H is balanced if its partite sets differ in size by at most one. A matching is a hypergraph consisting of disjoint edges. A hypergraph $H = (V, E)$ is a subhypergraph of $F = (V', E')$ if $V \subseteq V'$ and $E \subseteq E'$. Denote by $\mathrm{ex}(n, H)$ the maximum number of edges in an n-vertex r-uniform hypergraph containing no copy of H as subhypergraph. The density of an r-uniform hypergraph $H = (V, E)$ on n vertices is $d(H) = |E|/\binom{n}{r}$.

The **multicolor Ramsey number** for an r-uniform hypergraph H, denoted by $r_k(H)$, is the minimum n such that no matter how the edges of K_n^r are colored with k colors, there is a monochromatic copy of H.

We shall say that a hypergraph G is H-free if it does not contain a copy of H as subhypergraph and that a coloring of G is H-free if there is no monochromatic copy of H in G.

While there are a number of results in the literature about $r_k(H)$ when k is a small fixed number (see [6]), the case when H is fixed and k grows appears not to have been extensively studied. An important exception is the case when H is a matching and the Ramsey number is known exactly from the chromatic number of Kneser's graphs and hypergraphs, [25], [1]. Results for certain r-partite r-uniform H were obtained in [23] and for the loose cycle $H = C_3^3$ in [15].

In this paper we propose and start a systematic investigation of the growth rate of $r_k(H)$ for some fixed H as k grows. We state some general remarks about techniques to get upper bounds (using Turán numbers, reducing uniformity by taking traces) and lower bounds (using block designs or their approximations) for $r_k(H)$, some of them (Proposition 3, Lemmas 12, 13) are known among researchers working in this area. We apply them to 3-uniform hypergraphs in Theorems 4 - 11.

To determine the growth rate of $r_k(H)$ in general is known to be a very hard problem. For example, the best known bounds even for the smallest nontrivial graph case are $c^k < r_k(K_3) < c'k!$ for some positive constants c and c' (see Chung [7] and Erdős, Szekeres [12]). The only known non-trivial classical Ramsey number for cliques is $r_2(K_4^3) = 13$, due to McKay and Radziszkowski [28]. Define the tower function as follows: $t_1(n) = n$ and $t_{i+1}(n) = 2^{t_i(n)}$ for all $i \geq 1$. Erdős, Hajnal and Rado gave an upper bound for all cliques and a lower bound for only large cliques.

Theorem 1 (Erdős and Rado [11], Erdős, Hajnal and Rado [10])**.** *Let $s > r \geq 2$. There are positive integers $c = c(s, r) \leq 3(s - r)$, $s_0(r)$, and $c' = c'(s, r)$ such that*

$$t_r(c'k) < r_k(K_s^r) < t_r(ck \log k)$$

where the lower bound holds for $s \geq s_0(r)$.

It is worth noting that the lower bound in [10] was stated for the case when the number of colors, k, is fixed while r grows and the bound was only for large cliques. But the proof in [10] applies naturally to our case as well, when k grows and the other parameters are fixed. A recent result by Conlon, Fox and Sudakov, [5] implies a lower bound for cliques of smaller sizes, but still only for $s \geq 2r-1$. Duffus, Lefmann and Rödl [8] took another approach, using shift graphs, and proved a lower bound for cliques of all sizes $s > r$, but require k being fixed and $r \gg k$. Our first result is for cliques of all sizes, using a slight modification of the stepping-up lemma, due to Erdős and Hajnal (see Chapter 4.7 in [14]).

Theorem 2. *For any $s > r \geq 2$ and $k > r2^r$ we have*

$$r_k(K_s^r) > t_r\left(\frac{k}{2^r}\right).$$

An important proposition shows that $r_k(H)$ is polynomial in k if and only if H is r-partite.

Proposition 3. *Let $r \geq 2$ be fixed and H be a connected r-uniform hypergraph. Then $r_k(H)$ is polynomial in k if and only if H is r-partite. In particular, there are positive constants c and c', such that*

(i) If H is r-partite, then $r_k(H) = O(k^c)$

(ii) If H is not r-partite, then $r_k(H) \geq 2^{c'k}$.

Our other results are about $r_k(H)$ for 3-uniform H. Let $K_4^3 - e$ be a hypergraph obtained from K_4^3 by removing one edge. Our next theorem gives bounds on $r_k(K_4^3 - e)$ in terms of $r_k(K_3)$, showing that compared to the double-exponential bounds for K_4^3 from Theorems 1 and 2, the correct order of magnitude for $r_k(K_4^3 - e)$ is single-exponential. It is known that $r_2(K_4^3 - e) = 7$ and $13 \leq r_3(K_4^{(3)} - e) \leq 16$ ([31]).

Theorem 4. *For any $k \geq 2$,*

$$r_k(K_3) \leq r_{4k}(K_4^3 - e) \quad \text{and} \quad r_k(K_4^3 - e) \leq r_k(K_3) + 1.$$

Denote by F_5 the hypergraph with edges $\{abc, abd, cde\}$. We show that $r_k(F_5)$ behaves similarly to $r_k(K_3)$.

Theorem 5. *There is a positive constant c such that, for $k \geq 4$, $2^{ck} \leq r_k(F_5) \leq k!$.*

The simplest non-trivial triple systems have just two edges. The **kite** is a 3-uniform hypergraph with two edges sharing two vertices. The **bow** is a 3-uniform hypergraph with two edges sharing a single vertex. Since the Turán number of these hypergraphs and the existence of designs providing constructions are known, the next two results are rather straightforward, except the statements for small number of colors which are worked out in Section 7.

Theorem 6. *Let $r_k = r_k(bow)$. Then*

$$r_k = (1 + o(1))\sqrt{6k}.$$

If $k = \frac{\binom{n}{3}}{n}$ *and* $n \equiv 4, 8 \pmod{12}$*, then* $r_k = n + 1$*. Moreover,* $r_2 = 5, r_3 = r_4 = r_5 = 6$*,* $r_6 = 7$*,* $r_7 = r_8 = r_9 = r_{10} = 9$*,* $9 \leq r_{11} \leq r_{12} \leq r_{13} \leq r_{14} \leq 10, r_{15} = 11$.

Remark. Note that $r_k(bow)$ is the smallest multicolor Ramsey number among nontrivial 3-uniform hypergraphs since $r_k(H) \geq \min\{r_k(bow), r_k(kite), r_k(M)\}$, where M is a matching with 2 triples. Indeed, each nontrivial 3 uniform hypergraph contains at least two edges that form one of *bow*, *kite* or M, and by [25], $r_k(M) = k + 5$.

Theorem 7. *Let* $r_k = r_k(kite)$*. Then*

$$r_k = \begin{cases} k+1, & \textit{if } k \equiv 3 \textit{ (mod 6)} \\ k+1 \quad or \quad k+2, & \textit{if } k \equiv 4 \textit{ (mod 6)} \\ k+2, & \textit{if } k \equiv 0, 2 \textit{ (mod 6)} \\ k+3, & \textit{if } k \equiv 1, 5 \textit{ (mod 6)}, k \neq 5 \\ 6 & \textit{if } k = 5, \\ 5 & \textit{if } k = 4 \end{cases}$$

Let a, b be positive integers. Denote by $F(a, b)$ the 3-uniform hypergraph with vertex set $V = A \cup B$, $A \cap B = \emptyset$, $|A| = a, |B| = b$ and edge set consisting of all triples with one vertex in A and two vertices in B (for example, $F(2, 2)$ is the kite).

Proposition 8. *For any* $a \geq 2$*, we have*

$$k(a-1) < r_k(F(a, 2)) \leq k(a-1) + 3.$$

In general, $r_k(F(a, b))$ grows slower than double exponential in k and possibly faster than exponential in k. (Recall that Theorems 1 and 2 give double-exponential bounds.)

Theorem 9. *Given* $3 \leq a \leq b$*, we have, for positive constants* $c = c(a, b)$ *and* $c' = c(a, b)$

$$2^{c'k} < r_k(F(a, b)) < r_t(K_b) + m < 2^{ck^{a+1} \log k},$$

where $m = (a-1)k + 1$*, and* $t = k\binom{m}{a}$*.*

The **windmill** W with *center edge* abc is the hypergraph with six vertices and edges abc, abd, bce, acf. The proof of the lower bound in the next two theorems illustrates how to combine designs with a result of Pippenger and Spencer [30] about partitioning of hypergraphs into matchings.

Theorem 10.

$$(1 - o(1))3k \leq r_k(W) \leq 3k + 3.$$

The ideas giving the asymptotic of $r_k(W)$ can be also used for the **tight path** $P_3^3 = \{abc, bcd, cde\}$.

Theorem 11. *$2k(1 - o(1)) \le r_k(P_3^3) \le 2k + 3$ and the upper bound is sharp when $k = 2^{2m-1} - 1$.*

The rest of the paper will be organized as follows. In Section 2, we give some auxiliary results and prove Proposition 3. Theorems 2 - 11 will be proved in Sections 3-6. Section 7 gives the arguments and designs needed for Theorems 6 and 7 for small number of colors, Section 8 contains remarks, conjectures and problems.

The lower bounds on Ramsey numbers often based on block designs. A $t - (v, k, \lambda)$ **design** is a subset of $\binom{[v]}{k}$, called blocks, such that each t element subset of $[v]$ is contained in exactly λ blocks.

2 General bounds and auxiliary results

In this section we prove some general bounds on $r_k(H)$ and obtain some consequences including Proposition 3. Recall that the density of an r-uniform hypergraph F with n vertices and e edges is $d(F) = \frac{e}{\binom{n}{r}}$.

Lemma 12. *Let H be a fixed r-uniform hypergraph and F be an H-free r-uniform hypergraphs with n vertices, density $d(F) = d$. Then*

(i) $r_k(H) \le 1 + \max\{n : \lceil \binom{n}{r} / \mathrm{ex}(n, H) \rceil \le k\}$,
(ii) If $\binom{n}{r}(1 - d)^k < 1$ then $r_k(H) \ge n$.

Proof. (i) Consider an H-free coloring of K_n^r with k colors. Then each color class has at most $\mathrm{ex}(n, H)$ edges.

(ii) Consider k copies of hypergraph F obtained by mapping its vertices randomly to a given set V of n vertices. Here, we choose vertex permutations uniformly. Assign the edges of the ith copy of F color i, $i = 1, \dots, k$. If an edge belongs to several copies of F, assign the smallest available label. We claim that with positive probability, each edge of $K = \binom{V}{r}$ belongs to some copy of F. Indeed, the probability that a given edge of K uncovered is $(1 - d)^k$. Thus, the probability that there is an uncovered edge of K is at most $\binom{n}{r}(1 - d)^k < 1$. Therefore, with positive probability, all edges are covered and the resulting coloring of K contains no monochromatic copy of H. $\square$

Proof of Proposition 3. (i) The proposition follows from Lemma 12(i) by using the fact that $\mathrm{ex}(n, H) < n^{r-c}$ for some positive constant $c = c(H)$, when H is r-partite, see [9]. So, $k \ge \binom{n}{r} / \mathrm{ex}(n, F) \ge Cn^r / n^{r-c} = Cn^c$, for a constant $C = C(r)$. Thus $n \le C^{-1/c} k^{1/c}$.

(ii) Let H be non-r-partite. Apply Lemma 12(ii) with F being a complete r-uniform r-partite balanced hypergraph on $n = 2^{c'k}$ vertices (and $r|n$). Clearly H is not contained in F as a subgraph. Moreover, $d(F) \ge \frac{(n/r)^r}{\binom{n}{r}} > \frac{(n/r)^r}{(en/r)^r} = e^{-r}$. Hence for $k = c \log n$ and $c > e^r(r + 1)$,

$$\binom{n}{r}(1 - d)^k = \binom{n}{r}(1 - d)^{c \log n} < n^r e^{-cd \log n} = e^{(r-cd) \log n} < 1. \qquad \square$$

The *trace* of a 3-uniform hypergraph H at vertex v is the graph on vertex set $V(H)-\{v\}$ and with edge set $\{e-\{v\}: \ e \in H, \ v \in e\}$. A transversal of a hypergraph is a set of vertices non-trivially intersecting each edge.

Lemma 13. *Let H be a 3-uniform hypergraph with a single-vertex transversal $\{v\}$. Let G be a trace of H with respect to v. Then $r_k(H) \leq r_k(G)+1$.*

Proof. Given an H-free k-coloring c of $\binom{[n]}{3}$, let c' be the k-coloring of $\binom{[n-1]}{2}$ defined by $c'(ij) = c(ijn)$. Then c' has no monochromatic G and consequently $r_k(G) \geq r_k(H)-1$ as required. □

For lower bounds on $r_k(H)$ the following decomposition result is a useful tool (we shall use it in Section 6):

Theorem 14 (Pippenger and Spencer [30])**.** *Let r be fixed and D be sufficiently large. Let H be an r-uniform hypergraph with $d(v) = (1+o(1))D$ for every $v \in V(H)$ and codegree of $o(D)$ for every pair $\{u,v\} \subseteq V(H)$. Then $E(H)$ can be partitioned into $(1+o(1))D$ matchings.*

3 K_s^r for $s > r \geq 2$

In this section we prove Theorem 2 using a variant of the stepping-up lemma of Erdős and Hajnal.

Proof of Theorem 2. It suffices to prove the result for $s = r+1$ since $r_k(K_s^r) \geq r_k(K_{r+1}^r)$ for any $s > r$. We use induction on r to show that $r_k(K_{r+1}^r) \geq t_r(k/2^{r-2}-2r)$ for all $k \geq r2^r$. Since $k \geq r2^r$, we have $k/2^{r-2}-2r \geq k/2^r$ and the result follows.

The base case $r=2$ is given by $r_k(K_3) > 2^k > 2^{k-4} = t_2(k-4)$. Assume the result holds for some $r \geq 2$ and let $n = r_k(K_{r+1}^r)-1$. By the inductive hypothesis $n \geq t_r(k/2^{r-2}-2r)-1$. In the next paragraph, we will construct a K_{r+2}^{r+1}-free coloring $\psi : \binom{[2^n]}{r+1} \to [2k+2r-4]$. This shows that $r_{2k+2r-4}(K_{r+2}^{r+1}) \geq 1+2^n \geq \frac{1}{2}t_{r+1}(k/2^{r-2}-2r)$. Now suppose we are given $k' \geq (r+1)2^{r+1}$. Let $k = \lfloor (k'-2r+4)/2 \rfloor$. Then $k \geq k'/2-r+1 \geq r2^r$ and $k' \geq 2k+2r-4$. Therefore $r_{k'}(K_{r+2}^{r+1})$ is at least

$$r_{2k+2r-4}(K_{r+2}^{r+1}) \geq \frac{1}{2}t_{r+1}\left(\frac{k}{2^{r-2}}-2r\right) \geq \frac{1}{2}t_{r+1}\left(\frac{k'}{2^{r-1}}+\frac{1-r}{2^{r-2}}-2r\right) > t_{r+1}\left(\frac{k'}{2^{r-1}}-2(r+1)\right).$$

Now we shall construct a K_{r+2}^{r+1}-free coloring ψ of $\binom{[2^n]}{r+1}$ using the K_{r+1}^r-free coloring ϕ of $\binom{[n]}{r}$. Represent the elements of $[2^n]$ with 0-1-sequences on n coordinates. For a vertex u and integer i, we denote $u(i)$ the ith coordinate of u in this representation. Given two vertices $u, v \in [2^n]$, say that $u < v$ if $u(i) < v(i)$ and $u(j) = v(j)$ for $j < i$. Denote such an i by $f(uv)$. Given any $u_1 < \cdots < u_{r+1}$, let $f_i := f(u_i u_{i+1})$, for every $1 \leq i \leq r$. Observe crucially that $f_i \neq f_{i+1}$, for every $1 \leq i \leq r-1$.

We define coloring ψ as follows:

$$\psi(u_1...u_{r+1}) = \begin{cases} (\phi(f_1,...,f_r),1) & \text{if } (f_1,...,f_r) \text{ is an increasing sequence,} \\ (\phi(f_1,...,f_r),2) & \text{if } (f_1,...,f_r) \text{ is a decreasing sequence,} \\ (i,3) & \text{if } f_1 < f_2 < \cdots < f_i > f_{i+1}, 2 \le i \le r-1, \text{ for } r \ge 3, \\ (i,4) & \text{if } f_1 > f_2 > \cdots > f_i < f_{i+1}, 2 \le i \le r-1, \text{ for } r \ge 3. \end{cases}$$

Suppose to the contrary that there is a monochromatic copy of K^{r+1}_{r+2} under ψ on vertex set $U = \{u_1, ..., u_{r+2}\}$ with $u_1 < \cdots < u_{r+2}$. Without loss of generality, we distinguish two cases.

Case 1: The second coordinate of ψ on each $(r+1)$-tuple of U is 1.
Considering ψ on $u_1, ..., u_{r+1}$ and $u_2, ..., u_{r+2}$, this implies that $f_1 < f_2 < \cdots < f_r < f_{r+1}$. Let $F := \{f_1, ..., f_{r+1}\}$ and $U = \{u_1, \ldots, u_{r+2}\}$. We see that all r-element subsets of F have the same color. Thus a monochromatic K^{r+1}_{r+2} on U under ψ yields a monochromatics K^r_{r+1} on F under ϕ, a contradiction.

Case 2: Each $(r+1)$-tuple of U has color $(i,3)$ for some i with $2 \le i \le r-1$.
Then $\psi(u_1, ..., u_{r+1}) = (i,3)$ implies $f_i > f_{i+1}$. On the other hand, $\psi(u_2, ..., u_{r+2}) = (i,3)$ implies $f_i < f_{i+1}$, a contradiction.

If the second coordinate is 2 or 4 the arguments are almost identical to those in Case 1 or 2. $\square$

4 $K^3_4 - e$ and F_5

Notice that in contrast to the double-exponential growth for K^3_4, $r_k(K^3_4 - e)$ is single-exponential in the number of colors k. Indeed, since $K^3_4 - e$ is not 3-partite, Proposition 3 yields $r_k(K^3_4 - e) > 2^{ck}$. For the upper bound, one can use a variation of the classical Erdős-Rado pigeonhole argument to obtain $r_k(K^3_4 - e) < 2^{(k+1)\log k}$. We will, however, use a different approach to prove this fact, which also shows some connection between the multicolor Ramsey number of $K^3_4 - e$ and the multicolor Ramsey number of a triangle.

Proof of Theorem 4. For the lower bound, let $n = r_k(K_3) - 1$ and $\phi : \binom{[n]}{2} \to k$ be a triangle-free k-coloring of $\binom{[n]}{2}$. We will construct a $K^3_4 - e$-free coloring ψ of $\binom{[n]}{3}$ with $4k$ colors. This then would imply that $r_{4k}(K^3_4 - e) \ge n + 1 = r_k(K_3)$ as desired. Let ψ be the following coloring of the triples $i < j < k$. If P is a path with vertices i, j, k, denote by $\phi'(P)$ the color under ϕ of the edge in $\{i, j, k\}$ that is not in P. For such a path P, let the type of P, $t(P) = 1, 2$, or 3 if i, j or k is its center, respectively. If $\{i, j, k\}$ is a rainbow triangle, let $\psi(ijk) = (0, \phi(jk))$. If $\{i, j, k\}$ induces a monochromatic path P, let $\psi(ijk) = (t(P), \phi'(P))$.

Suppose there is a monochromatic copy $K = \{abc, abd, acd\}$ of $K^3_4 - e$, we will show a contradiction when the first coordinate is 0, namely all three triples $\{abc, abd, acd\}$ span rainbow triangles under

ϕ. The cases when the first coordinate is $1, 2$ or 3, can be proved using a similar argument. Notice that when the first coordinate is 0, by the definition of ψ, the color of a triple depends on the color, under ϕ, of the edge spanned by the two largest elements in that triple. Since b, c, d play a symmetric role, we can assume that $b < c < d$. If a is the smallest, then $\psi(abc) = \psi(abd) = \psi(acd)$ implies $\phi(bc) = \phi(bd) = \phi(cd)$, i.e. bcd is monochromatic under ϕ. Thus b is the smallest. But then $\psi(abc) = \psi(abd)$ implies $\phi(ac) = \phi(ad)$, which means acd is not a rainbow triangle under ϕ, a contradiction.

For the upper bound, simply notice that $K_4^3 - e = \{abc, abd, acd\}$ has a single vertex transversal $\{a\}$, and the trace of a is a triangle on $\{b, c, d\}$. Thus the upper bound follows from Lemma 13. The case with 2 colors is treated in Section 7. □

Proof of Theorem 5. The lower bound follows from Proposition 3(ii), since F_5 is not 3-partite. The upper bound comes by induction with basis $k = 4$. Suppose that the edges of K_{24}^3 with vertex set V can be 4-colored so that there is no monochromatic F_5. There are 22 triples uvx containing a fixed pair uv. Assume that uvx_1, uvx_2 are red triangles. Then x_1x_2y cannot be red for $y \in Y = V - \{u, v, x_1, x_2\}$. Thus we have a set S, $S \subseteq Y$, $|S| \geq \lceil (|V| - 4)/3 \rceil = 7$ and x_1x_2y are blue triples for all $y \in S$. Therefore, no triple in S is colored blue, and thus $\binom{S}{3}$ uses $k - 1 = 3$ colors. But $r_3(F_5) = 7$ (and $r_2(F_5) = 6$), see Section 10.

The inductive step is simply repeating the argument above in general. Suppose we already know $r_k(F_5) \leq k!$ for some $k \geq 4$ and we have a K_n^3 with a F_5-free $(k+1)$-coloring. Selecting u, v, x_1, x_2 as above and applying the same argument, we get $n - 4 \leq k(k! - 1) < (k + 1)! - k$, thus $n \leq (k + 1)! - k + 4 \leq (k + 1)!$. This implies $r_{k+1}(F_5) \leq (k + 1)!$. □

Remark. The above results slightly suggests that $r_k(F_5) \leq r_k(K_3)$ might hold. Although the bound $r_k(F_5) \leq k!$ in Theorem 5 can be improved slightly, this improvement still does not show that $r_k(F_5) \leq r_k(K_3)$.

5 Bow, Kite, $F(a, b)$

Lower bounds of $r_k(bow)$ follow from the existence of resolvable designs. A $3 - (n, 4, 1)$ design is a set of 4-element subsets (blocks) of an n-element set V such that each 3-element subset of V is in precisely one block. A $3 - (n, 4, 1)$ design is called *resolvable* if its blocks can be grouped so that each group (parallel class) gives a partition of V.

Proof of Theorem 6. When $n \equiv 4, 8 \pmod{12}$, $k = \frac{\binom{n}{3}}{n}$, $\text{ex}(n, bow) = n$, thus Lemma 12 (i) gives $r_k \leq n + 1$. This is sharp, since resolvable $3 - (n, 4, 1)$ designs exist if $n \equiv 4, 8 \pmod{12}$, see [18, 19], and [21]. The statement $r_k(bow) \approx \sqrt{6k}$ follows from considering that design for the largest n, $n \equiv 4, 8 \pmod{12}$, $k \geq \frac{\binom{n}{3}}{n}$ (for the lower bound) and applying the Lemma 12(i) for the upper bound. The statements about the small values are proved in Section 7. □

Proof of Theorem 7. Let $H = F(2, 2)$ be the kite. Then $\mathrm{ex}(n, H)$ corresponds to the maximum number of triples on n elements such that any two triples intersect in at most one element, i.e. the maximum number of edges in a linear 3-uniform hypergraph. A well-known result of Schönheim [32] and others (the cases $n \equiv 0, 1, 2, 3 \pmod 6$ go back even to Kirkman [22]) is $\mathrm{ex}(n, H) = \left\lfloor \frac{n}{3} \left\lfloor \frac{n-1}{2} \right\rfloor \right\rfloor - \epsilon$, where $\epsilon = 1$ for $n \equiv 5 \pmod 6$, otherwise $\epsilon = 0$. Lemma 12(i) gives, after some calculations, the upper bounds.

The lower bound for the cases $k \equiv 3, 4 \pmod 6$ is easy. Given $K_n^3 = (V, E)$, consider $V = Z_n$ and color triple ijk with color $i + j + k \pmod n$. Clearly this coloring yields no monochromatic H, hence $r_k(H) > k$.

For the cases $k \equiv 0, 1, 2, 5 \pmod 6$ the (difficult) constructions of J. X. Lu [26, 27] finished by Teirlinck [33] are needed: for $n > 7, n \equiv 1, 3 \pmod 6$, K_n^3 can be partitioned into $n - 2$ Steiner triple systems (called a *large set* of STS).

Indeed, for $k \equiv 0, 2 \pmod 6$ we need a kite-free k-coloring of K_{k+1}^3 i.e. $(n-1)$-coloring of K_n^3 when $n \equiv 1, 3 \pmod 6$. This can be done even with $n - 2$ colors according to the cited result of Lu. However, the case $k = 6$ is exceptional because Lu's theorem does not hold for $n = 7$. Nevertheless, there is a kite-free 6-coloring of K_7^3 as shown in Proposition 15. Similarly, for $k \equiv 1, 5 \pmod 6$ we need a kite-free k-coloring of K_{k+2}^3 i.e. $(n - 2)$-coloring of K_n^3 when $n \equiv 3, 1 \pmod 6$. This is provided by Lu's theorem, apart from the case $k \equiv 5$ $(n = 7)$ which is indeed exceptional, in Proposition 15 we prove that $r_5(\mathit{kite}) = 6$ (together with the case $k = 4$). □

Proof of Proposition 8. In an $F(a, 2)$-free coloring of K_n^3 any pair of vertices is in at most $a - 1$ edges of the same color. Thus $n \leq 2 + k(a - 1)$, proving the upper bound. (One can also use Lemma 13 and the multicolor Ramsey number for stars (see [3]): $r_k(K_{1,a}) \leq k(a - 1) + 2$.)

For the lower bound, set $n = k(a - 1)$ and consider $K_n^3 = (V, E)$ with $V = Z_n$. Color a each edge with the sum of its vertices mod k. Then a monochromatic copy of $F(a, 2)$ would require that for some $y, z \in V$, $y + z + x_1, ..., y + z + x_a$ are all equal $\pmod k$ i.e. we have a different positive x_s, all equal $\pmod k$, which is impossible. Hence $r_k(F(a, 2)) > k(a - 1)$. □

Proof of Theorem 9. For the upper bound, let $N = r_t(K_b) + m$. Consider a k-coloring ϕ of the triples of K_N. Fix a set S of m vertices and define a t-coloring c on the pairs of the remaining $N - m$ vertices as follows. Let $c(xy) = (\phi(xys_i), s_1, s_2, ..., s_a)$, where $\phi(xys_i)$ is the majority color on triples containing x and y, and $s_1, s_2, ..., s_a \in S$ is the lexicographically first a-tuple in S such that $\phi(xys_i) = \phi(xys_j)$ for every $1 \leq j \leq a$ (by the choice of m there is such an a-tuple). Since c is a t-coloring of a complete graph on $N - m = r_t(K_b)$ vertices, there is monochromatic K_b in c, which gives a monochromatic $F(a, b)$ in ϕ.

A lower bound for $r_k(F(a, b))$ is obtained from Proposition 3 (i) since $F(a, b)$ is not 3-partite, for $b \geq 3$. □

6 Windmill and tight path

Proof of Theorem 10. To prove the lower bound, let S be a $3-(n,5,1)$ design. The existence of such designs are known for infinitely many n, for example for $n = 4^s + 1, s \geq 2$ [20], see also [29]. Construct an auxiliary 10-uniform hypergraph H where $V(H)$ is the set of $\binom{n}{2}$ pairs in $V(S)$, and ten of these pairs form an edge of H if and only if they are the ten pairs in a block of S. Since every pair in $V(S)$ is in exactly $(n-2)/3$ blocks of S, H is an $(n-2)/3$-regular hypergraph. On the other hand, the codegree of any two vertices in H is at most one. Indeed, any two vertices in H (two pairs in $V(S)$) contain at least three vertices in $V(S)$, and they can be in at most one block of S. With large enough n, and with $r = 10, D = n/3$, the conditions of Theorem 14 hold so we can decompose $E(H)$ into $m = (1+o(1))n/3$ matchings $M_i, i = 1, 2, \ldots, m$. Each M_i corresponds to a subset of blocks S_i of S and any two blocks in S_i share at most one element in $V(S)$. The set of triples covered by the blocks of any S_i form a W-free triple system (the center edge of a windmill W in a block $B \in S_i$ would force the other three edges of W to B). Thus K_n^3 is decomposed into $m = (1+o(1))n/3$ W-free triple systems, showing $r_k(W) \geq (1-o(1))3k$.

The upper bound follows from Lemma 12(i), applying a special case of a theorem of Frankl and Füredi ([13], Theorem 3.8): $\mathrm{ex}(n,W) \leq \binom{n}{2}$. (In fact, this is sharp for every $n \equiv 1, 5 \pmod{20}$. To see that, consider a $2-(n,5,1)$ design, its existence is proved by Hanani [16, 17]. The number of blocks is $\binom{n}{2}/10$, place 10 triples inside each block of S. The resulting triple system, H, has $\binom{n}{2}$ triples and is W-free.) □

Proof of Theorem 11. Observe that the trace of P_3^3 at its transversal vertex is P_4, the path on four vertices. Apply Lemma 13, $r_k(P_3^3) \leq r_k(P_4) + 1 \leq 2k + 3$ ([31]).

For the lower bound we start with a $3-(n,4,1)$ design F (already used in the proof of Theorem 6) and follow the construction in the proof of Theorem 10. Consider the 6-uniform hypergraph H with vertex set being the set of pairs of vertices of F and edges formed by the sets of pairs within the blocks of F. The degree of any vertex in H is $d = (n-2)/2$, the codegree of any pair of vertices is at most one, so the conditions for Pippenger-Spencer Theorem are satisfied, giving a decomposition of H into $(1+o(1))d = (1+o(1))n/2$ matchings, M_i. Each M_i corresponds to a set F_i of blocks of F, intersecting each other in at most one element. Let T_i be the set of triples covered by the blocks of F_i. The T_i-s provide the required P_3^3-free coloring of K_n^3 with $(1+o(1))n/2$ colors. To see that the upper bound is sharp when $k = 2^{2m-1} - 1$, i.e $2k + 2 = 2^{2m}$, one can use a result of R.D.Baker [2]: there exists a $3-(2k+2,4,1)$ design which can be partitioned into k $2-(2k+2,4,1)$ designs. □

7 Ramsey numbers of bow and kite for small number of colors

The following proposition determines the small undecided cases from Theorem 7. A hypergraph is linear if every two edges share at most one vertex.

Proposition 15. $r_4(kite) = 5, r_5(kite) = 6, r_6(kite) = 8$.

Proof. It is obvious that $r_4(kite) > 4$. The fact that $r_4(kite) \leq 5$ follows by observing that any 4-coloring of the edges of K_5^3 contains three edges of the same color.

Coloring the triple ijk, $1 \leq i < j < k \leq 5$ by color $i + j + k \pmod 5$ gives $r_5(kite) > 5$. To show that $r_5(kite) \leq 6$, we need the result of Cayley [4], stating that the maximum number of pairwise disjoint Fano planes in K_7^3 is 2. Suppose K_6^3 on vertex set V is 5-colored so that each color class i is a linear hypergraph P_i. Since the average number of edges in a color class is four and no linear hypergraphs on 6 vertices can have more than four edges, it follows that each P_i must be a Pasch configuration. Therefore the pairs uncovered by the triples of P_i form a matching M_i in the complete graph on V. The M_i-s must form a factorization on V otherwise some pair in V would be covered by at most three P_i-s instead of the required four. These P_i-s can be extended by a new vertex to a decomposition of K_7^3 into five Fano planes, contradicting Cayley's theorem stated above.

The upper bound $r_6(kite) \leq 8$ is already proved (see the proof of Theorem 7). For the lower bound we need a partition of K_7^3 into six linear hypergraphs $F_1, \ldots, F_4, F_6, F_7$ on a vertex set $V = [7]$, see Figure 1. Let F_1, F_2 be the two Fano planes generated by shifts of $124, 134 \pmod 7$. The next two sets F_3 and F_4 are isomorphic to a Fano plane from which one line is deleted:

$$F_3 = \{135, 167, 236, 257, 347, 456\}, \quad F_4 = \{123, 146, 247, 256, 345, 367\},$$

F_6 is a Fano plane from which two lines are deleted:

$$F_6 = \{127, 136, 145, 246, 567\},$$

and F_7 is a Pasch configuration:

$$F_7 = \{125, 147, 234, 357\}.$$

□

Proposition 16. *Set* $r_k = r_k(bow)$, *then* $r_1 = r_2 = 5, r_3 = r_4 = r_5 = 6,\ r_6 = 7, r_7 = r_8 = r_9 = r_{10} = 9,\ 9 \leq r_{11} \leq r_{12} \leq r_{13} \leq r_{14} \leq 10, r_{15} = 11$.

Proof. All upper bounds but one are obtained from Lemma 12(i). The exceptional case is when Lemma 12(i) gives $r_5(bow) \leq 7$. Here we improve it as follows. Suppose K_6^3 is 5-colored without monochromatic bow. One can easily see that each color class is either a K_4^3 (type A) or four triples pairwise intersecting in the same base pair (type B). There are at most three type A colors. The base pairs for different type B colors must be vertex disjoint. Thus there are at least two type A color classes, w.l.o.g. $abcd, cdef$. But then only the base pairs ae, af, be, bf are available for type B colors. Therefore we have two type B and three type A colors, the third is the K_4^3 spanned by $abef$. Now there is no base pair available for type B color classes since every pair of vertices is covered by a type A K_4^3.

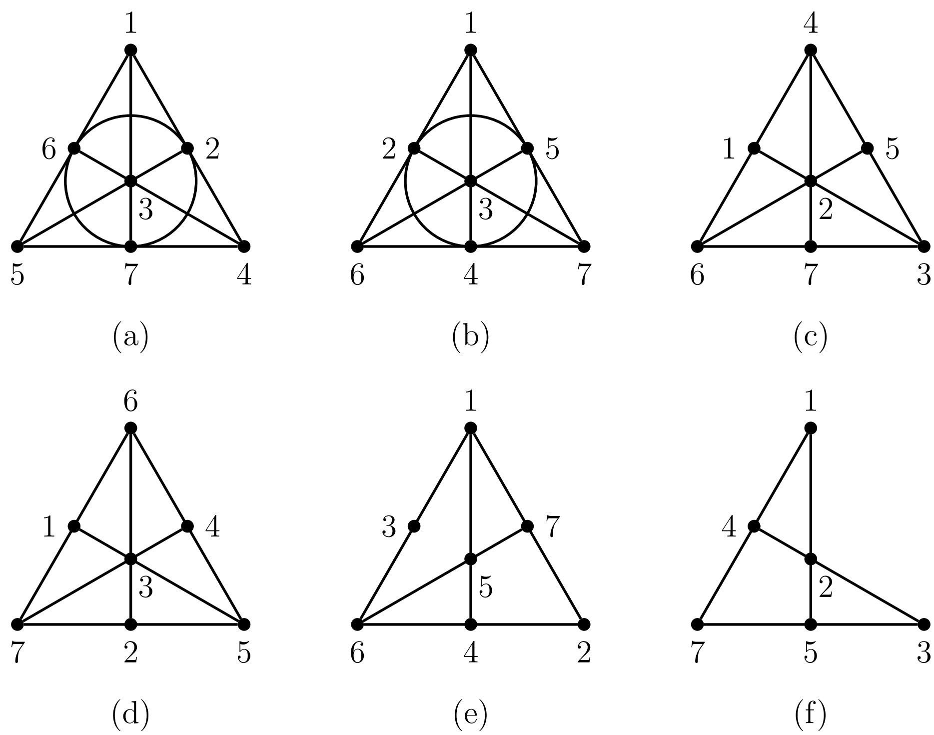

Figure 1: Partition of K_7^3 into two Fano, two Fano$-e$, Fano$-2e$, Pasch

Lower bounds should be exhibited for $r_1, r_3, r_6, r_7, r_{15}$ only. Coloring all triples of K_4^3 with the same color, $r_1 > 4$ follows. Coloring the triples of $\{1,2,3,4\}$ with color 1, the triples $125, 135, 235$ with color 2, the triples $145, 245, 345$ with color 3, $r_3 > 5$ follows. Then $r_6 > 6$ comes from the following 6-coloring with color classes $\binom{\{1,2,3,4\}}{3}$, $\binom{\{3,4,5,6\}}{3}$, $\binom{\{1,4,5,6\}}{3} - \{4,5,6\}$, $\binom{\{2,4,5,6\}}{3} - \{4,5,6\}$, $\binom{\{1,2,3,5\}}{3} - \{1,2,3\}$, $\binom{\{1,2,3,6\}}{3} - \{1,2,3\}$. The 7-coloring of K_8^3 is the 7 parallel classes of the unique $3-(8,4,1)$ design. Finally, the 15-coloring of K_{10}^3 comes from the unique $3-(10,4,1)$ design whose 30 blocks can be partitioned into 15 disjoint pairs. □

8 Concluding remarks

We determined, for 3-uniform hypergraphs, r_k ranges from $\sqrt{k}$ to double exponential in k, and showed a jump in r_k when H changes from r-partite to non-r-partite. This leads to the following question.

Problem 17. *For which* 3*-uniform hypergraphs* F*, is* $r_k(F)$ *double exponential? Are there other jumps that the Ramsey function* r_k *exhibits?*

The ramsey-numbers $r_k(bow), r_k(kite)$ are closely connected to block designs. In case of the kite the only uncertainty is whether $r_k(kite)$ is $k+1$ or $k+2$ when $k \equiv 4 \pmod 6$. This leads to the following problem.

Problem 18. *Suppose $n \equiv 5$ (mod 6). Is it possible to partition the triples of an n-element set into $n-1$ partial triple systems, i.e. into parts so that distinct triples in each part intersect in at most one vertex? By Theorem 7, this is not possible for $n = 5$ but perhaps for large enough n (possibly for $n \geq 11$) such partitions exist.*

In case of the bow, the problems related to sharper bounds of $r_k(bow)$ are not purely design theoretic, since color classes can be star components as well. We state just one of those problems.

Problem 19. *Suppose $n \equiv 6, 10$ (mod 12). Is it possible to partition the triples of an n-element set into $\frac{n(n-1)}{2}$ classes so that each class is the union of some disjoint K_4^3-s and at most one star component? (Any color class has $n-2$ triples.) For $n = 6$ there is no solution.*

Concerning $r_k(K_3 - e)$ the most challenging (perhaps difficult) problem is to decrease the upper bound of Theorem 4 by one.

Problem 20. $r_k(K_4^3 - e) < r_k(K_3) + 1$ *for every $k \geq 3$?*

A challenging open problem is to improve the estimates of $r_k(P)$ (and/or $ex(n, P)$) where P is the *Pasch configuration* with edges $\{abc, bde, cef, adf\}$. (It can be obtained from the Fano plane by deleting a vertex.) Presently only the following is known.

Proposition 21. *For positive constants c, c',* $c\left(\frac{k}{\log k}\right)^2 < r_k(P) < c'k^4$.

Proof. The lower bound is based on the following P-free hypergraph, showing that $\mathrm{ex}(n, P) = \Omega(n^{5/2})$, [24]. Take an incidence graph G of a projective plane with n points and n lines. It has $\Omega(n^{3/2})$ edges. Add n new vertices $x_1, ..., x_n$ and add all triples of the form $x_i \cup e$, where e is an edge of G. The resulting 3-uniform hypergraph, call it H, has $3n$ vertices and $\Omega(n^{5/2})$ edges.

Notice that the edge-density of H is $d(H) = cn^{-1/2}$ for some constant $c > 0$. From Lemma 12(ii) we see that there is a coloring of K_n^3 with $(c'n^{1/2} \log n)$ colors and no monochromatic P. Thus $r_k(P) > n$ with $k = c'n^{1/2} \log n$. Expressing n in terms of k gives the desired lower bound.

The upper bound follows from Lemma 12(i) and the fact that $\mathrm{ex}(n, P) = O(n^{11/4})$ [24]. This is based on the claim that $\mathrm{ex}(n, K(2, 2, 2)) = O(n^{11/4})$ proved by Erdős [9], where $K(2, 2, 2)$ is the complete 3-partite 3-uniform hypergraph with two vertices in each part. □

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10 Appendix - for arXiv and for referees

Proposition 22. $r_2(F_5) = 6$.

Proof. The lower bound is obvious, color triples of K_5^3 containing a fixed vertex with color 1 and other triples by color 2. For the upper bound, consider a 2-colored K_6^3 on vertex set $\{1, 2, 3, 4, 5, 6\}$ and its 2-colored trace $K = K_5^2$ with respect to vertex 6. There is a monochromatic, say red odd cycle C in $V(K) - \{6\}$. If $C = 1, 2, 3, 1$ then either there is a red triple in K with two vertices on C and one vertex not in C or all such triples are blue. The former gives a red, the latter a blue F_5. If $C = 1, 2, 3, 4, 5, 1$ then either there is a red triple with vertices non-consecutive on C or all the five such triples are blue. Again, the former gives a red, the latter a blue F_5. □

Theorem 23. $r_3(F_5) = 7$.

Proof. For the lower bound, color the triples of K_6^3 containing v with color 1, color uncolored triples containing vertex $w \neq v$ with color 2 and color all other edges with color 3.

To prove the upper bound, call a graph G *nice* if for every triple $T = \{v_1, v_2, v_3\}$ of vertices at least one of the following holds:
1. There are two vertex disjoint edges of G, such that one of them is in T and the other meets T.
2. There is a path of length two in G connecting two vertices of T with midpoint not in T.

Observation 24. *If H is an F_5-free 3-uniform hypergraph, such that the trace of v for a vertex v is a nice graph, then all edges of H within $V(G) \cup \{v\}$ contain v.*

Indeed, otherwise from the definition of a nice graph we find F_5 in H. Thus finding a large nice subgraph in a trace one can reduce the number of colors. More generally, a graph is i-nice if the property holds for all but at most i triples of vertices.

We need a lemma on 6-vertex graphs. Since its proof is routine but lengthy, we state it without proof.

Lemma 25. *Suppose G has six vertices. If $|E(G)| \geq 9$ then G is nice. If $|E(G)| = 8$ then G is 1-nice, if $|E(G)| = 7$ then G is 2-nice. If $|E(G)| = 6$ then G is 5-nice, except in one case, when G is $K_{2,3}$ plus an isolated vertex (in this case it is 6-nice).*

With these preparations we are ready to prove the upper bound. The majority color, say red in a 3-colored K_7^3, has at least 12 edges. Some vertex v has red degree at least 6. Let G be the trace of a red hypergraph at v. We get a contradiction from Lemma 25 (and from the fact that we have 12 edges) except when G has exactly six edges and the trace is $K_{2,3} + w$. This case implies that the red color class has 12 edges forming $K_{2,2,3}$, a complete 3-partite hypergraph with parts of sizes 2, 2, and 3. However, among the $35 - 12 = 23$ edges of other colors, one color, say blue, has at least 12 edges. Repeating the argument for the blue hypergraph, we conclude that the blue hypergraph is also a $K_{2,2,3}$. However, as one can easily check, there is no way to place two edge disjoint $K_{2,2,3}$-s on 7 vertices. □