Lecture 9

Recall:

Schwartz-Zippel Lem

f non-zero n-var deg-d, $d<p$, on $\mathbb{F}_p^n$

$\Rightarrow f$ has $\leq d \cdot p^{n-1}$ roots.

- testing $f=g$? randomised alg.
- testing PM. G has$_\wedge^{\text{NO}}$ PM $\Leftrightarrow$ det(Edmonds mat) is 0-polyn.

Def: $K \subseteq \mathbb{F}_p^n$ Kakeya set if it has a line in every direction

Thm [Dvir 09] $\forall\ K \subseteq \mathbb{F}_p^n$ Kakeya

$$\Rightarrow |K| \geq \binom{p+n-1}{n}$$

Rmk: Think of n fixed, $p \to \infty$, $\binom{p+n-1}{n} \sim p^n/n!$

In constrast to the Euclidean case, Kakeya set in finite field is dense.

- Let's start w/ giving a lower bd on the dim. of low-deg multivar. polyn on $\mathbb{F}_p^n$.

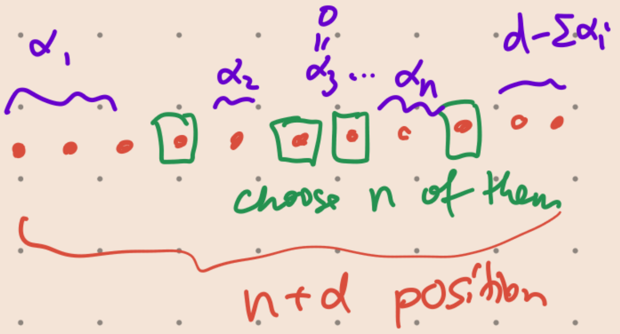
Lem 1: $\forall\ A \subseteq \mathbb{F}_p^n$ w/ $|A| < \binom{d+n}{d}$

$\Rightarrow \exists$ non-zero deg-d polyn f on $\mathbb{F}_p^n$ vanishing on A.

Pf: • n-var. deg-d f = linear combination of deg $\le d$ monomials in $x_1, \dots, x_n$.

$$f(x_1,\dots,x_n) = \sum_{\substack{\alpha_1+\dots+\alpha_n \le d \\ \alpha_i \ge 0}} c_\alpha x_1^{\alpha_1} \cdots x_n^{\alpha_n},$$

• # such monomials $= \binom{d+n}{d}$ = # choices of deg α



= # ways to put d identical balls into $n+1$ distinct boxes

• treat each c_α as unknowns, each $a \in A$ as constraints $\Rightarrow$ linear sys. w/ # unknowns > # constr.

$\Rightarrow \exists$ non-zero sol^n. ∎

<u>Lem 2</u>. If a non-zero deg-d ($d<p$) polyn f vanishes on a Kakeya set K, then its deg-d part f_d is a non-zero polyn vanishing everywhere on $\mathbb{F}_p^n$.

In other words, if a polyn. of deg $<p$ vanishes on K, then it must be 0-polyn.

<u>Pf</u>: • $\forall$ direction $z \in \mathbb{F}_p^n$, $z \neq 0$, K contains a line $L = \{a + tz : t \in \mathbb{F}_p\}$ for some $a \in \mathbb{F}_p^n$.

• Recall that the restriction of f on L

f_L is deg-d polyn. w/ leading coeff. $f_d(z)$

As f vanishes on K, f_L vanishes on L.

but $\deg f_L = d < p = |L|$

$\Rightarrow f_L$ is 0-polyn. $\Rightarrow f_d(z) = 0$

- Obviously $f_d(0) = 0 \Rightarrow f_d$ vanishes on $\mathbb{F}_p^n$. ▨

Pf (Dvir's bound)

- Take K Kakeya, supp. $|K| < \binom{n+p-1}{n}$
- Lem 1 $\Rightarrow \exists$ non-zero polyn f $\deg \leq (p-1)$ vanishing on K. $\lightning$ Lem 2. ▨.

§3.1 tight bound.

- Dvir : lower bdd $\geq 1/n! \cdot p^n$
- Saraf-Sudan (Dvir-Kopparty) Upp bdd : $\exists$ Kakeya set of size $2^{-n+1} \cdot p^n + O_n(p^{n-1})$
- Dvir-Kopparty-Saraf-Sudan : lower bd $\geq (2-o(1))^{-n} \cdot p^n$.
- Bukh-Chao $\geq (2-1/p)^{-n+1} \cdot p^n$.

They do so by considering a larger vector sp. of polyn. and subsp. that vanishes on a Kakeya to high order.

Dvir: Consider vect. sp. of deg $<p$ polyn. and show that subsp. vanishing on K must be trivial.

- $U = \left\{ \sum_{\substack{\alpha_1+\dots+\alpha_n<p \\ \alpha_i \geq 0}} c_\alpha x^\alpha : c_\alpha \in \mathbb{F}_p \right\}$ $\Leftrightarrow$ $\forall f \in U$ vanishes on K $f = 0$-polyn

- $U' \leq U$ vanishes on $K \overset{\text{Lem2}}{\Longrightarrow}$ U' is trivial!

- $|K| \geq \operatorname{codim} U' = \dim U = \binom{n+p-1}{n}$ ■

Bukh-Chao dim $n=3$ K Kakeya in $\mathbb{F}_p^3 \Rightarrow |K| \geq \frac{1}{4}(p^3+p^2)$

- $A = \left\{ (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{Z}_{\geq 0}^3 : \begin{array}{l} \alpha_1+\alpha_2+\alpha_3 < 2p \\ \alpha_1, \alpha_2 < p \end{array} \right\}$

- $V = \left\{ \sum_{\alpha \in A} c_\alpha x^\alpha : c_\alpha \in \mathbb{F}_p \right\}$.

Rmk: Considering V has some similarity to Green's twist on corner-free set.

Def: A poly f on $\mathbb{F}_p^3$ vanishes at $z \in \mathbb{F}_p^3$ to order 2 if $f(z) = 0$ & $\nabla f(z) = 0$ $\left(\begin{array}{l} \frac{\partial f}{\partial x_1}(z) = 0 \\ \frac{\partial f}{\partial x_2}(z) = 0 \\ \frac{\partial f}{\partial x_3}(z) = 0 \end{array} \right.$.

Lem3 (c.f. Lem2)
$\forall f \in V$, if f vanishes to order 2 on a Kakeya set $K \Rightarrow f$ is 0-polyn.

Pf [B-C] • Let $V' \leq V$ subsp. vanishing to order 2 on K

$\Rightarrow \operatorname{codim} V' \leq 4|K|$

- Lem 3 $\Rightarrow$ V' trivial
- $4|K| \geq \operatorname{codim} V' = \dim V = |A|$

$$= \sum_{\alpha_1,\alpha_2=0}^{p-1} (2p-\alpha_1-\alpha_2) = p^3+p^2 \qquad \blacksquare$$

<u>Pf</u> (Lem 3) • Supp. f non-zero $\in V$ deg d vanishing on K to order 2 $(d<2p)$.

$\Rightarrow f_d$ is non-zero.

- $\forall z \neq 0$ take a line $L = \{a + tz : t \in \mathbb{F}_p\} \subseteq K$
- Consider f_L • f_L vanishes to order 2 on L

but $\deg f_L = d < 2p = 2|L|$

$\Rightarrow f_L$ is 0-polyn.

$\Rightarrow$ so is its leading coeff. $f_d(z) = 0$

$\Rightarrow f_d$ vanishes on $\mathbb{F}_p^3$.

<u>WTS</u> f_d = 0-polyn.

- Let $g(x_1, x_2) = f_d(x_1, x_2, 1)$

$\Rightarrow g$ vanishes on $\mathbb{F}_p^2$

- Think of g as a polyn in x_1 of deg $<p$ (as $\alpha_1 < p$)

$$g(x_1, x_2) = \sum_{i=0}^{p-1} x_1^i \cdot \underbrace{g_i(x_2)}_{\deg < p \text{ as } \alpha_2 < p}$$

- g vanishes on $\mathbb{F}_p^2 \Rightarrow g_i$ vanishes on $\mathbb{F}_p$

  but $\deg g_i < p \Rightarrow g_i = 0$-polyn.

  $\Rightarrow g = 0$-polyn.

- f_d homogeneous, $f_d(x_1, x_2, 1) = g(x_1, x_2)$

  $\Rightarrow f_d$ is 0-polyn. $\blacksquare$