Lecture 7

Thm 1 [Kahn–Zhao]

$\forall$ n-vx d-reg G, $i(G) \le i(K_{d,d})^{n/2d}$

weighted extension

Thm 2 (DJPR) $\forall$ d-reg G $\forall \lambda > 0$

$$\Rightarrow \quad \bar{\alpha}_G(\lambda) \le \bar{\alpha}_{K_{d,d}}(\lambda) = \frac{\lambda(1+\lambda)^{d-1}}{2(1+\lambda)^d - 1}$$

Exer Show Thm 2 $\Rightarrow$ Thm 1.

Idea Draw $\begin{cases} I \sim_{hc(\lambda)} G \\ v \sim V(G) \end{cases}$ independently

Def r.v. $X = X(I, v) = |I \cap N(v)|$ = # occupied neighbr.

Write, for $k \in \{0, 1, \dots, d\}$, $p_k = \Pr(X = k)$

- Again bound $\bar{\alpha}_G(\lambda)$ in two ways

only when v suitable i.e. $X = 0$

looking at $\begin{cases} 1) \text{ if } v \in I & \cdots\cdots p_0 \\ 2) \; X \; (N(v) \cap I) & \cdots\cdots p_1, \dots, p_d \end{cases}$

Goal:

Max $\bar{\alpha}_G(\lambda)$ via a linear prog. (involving $p_0, p_1, \dots, p_d$)

Pf: 1) $\bar{\alpha}_G(\lambda) = \frac{1}{|G|} \sum_{u \in V(G)} \Pr(u \in I) = \Pr(v \in I)$

$$\frac{\mathbb{E}|I|}{|G|} = \Pr(\{v \in I\} \cap \{X = 0\}) = \Pr(v \in I \mid X = 0) \cdot \underbrace{\Pr(X = 0)}_{= p_0}$$

Recall $\Pr(v \in I \mid X=0) = \Pr(v \in I \mid v \text{ suitable})$

$$= \frac{\lambda}{1+\lambda}$$

$$\Rightarrow \quad \bar{\alpha}_G(\lambda) = \frac{\lambda}{1+\lambda} \cdot p_0$$

2) $$\bar{\alpha}_G(\lambda) = \frac{1}{|G|} \sum_{u \in V(G)} \frac{1}{d} \left| N(u) \cap I \right| = \frac{\mathbb{E}X}{d}$$

$$= \frac{1}{d}\left(p_1 + 2 \cdot p_2 + \dots + d\, p_d\right)$$

C2

So $$\bar{\alpha}_G(\lambda) = \boxed{\frac{\lambda}{1+\lambda} p_0 = \frac{1}{d}\left(p_1 + 2p_2 + \dots + d\,p_d\right)}$$

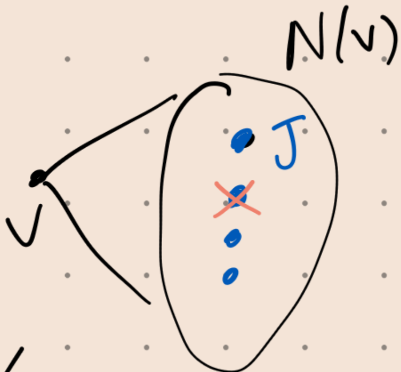
Goal Max p_0 s.t.

C1 $p_0 + p_1 + \dots + p_d = 1$

C2

C3 $\forall\ 2 \le k \le d, \quad (d-k+1) \cdot \lambda \cdot p_{k-1} \ge k\, p_k$.

Pf (C3). For an outcome (of $X=k$) J w/ $|J \cap N(v)| = k$



$\Rightarrow$ k ways from J to get J' w/ $|J' \cap N(v)| = k-1$.

- Recall $\lambda \Pr(J') = \Pr(J)$
- J' can be obtained from $\le d-k+1$ many other

choices of J w/ $|J \cap N(v)| = k$. ☒

- $\overline{\alpha_G}(\lambda) = \frac{\lambda}{1+\lambda} \cdot p_0$

  Suffices to show $\quad p_0 \le \frac{(1+\lambda)^d}{2(1+\lambda)^d - 1}$ .... (♡)

- if a choice of $(p_0, \dots, p_d)$ max p_0

  $\Rightarrow$ all eq hold in C3

  Supp not, say $(d-k+1)\lambda p_{k-1} > k p_k$

  then we can
  - increase p_0 by small $\varepsilon > 0$
  - move some mass (funct. of ε) from $p_{k-1} \to p_k$
  - other p_i fixed

  $\Rightarrow$ p_0 is not max. ↯

Now that eq in C3 holds,

we have a sys. lin. eq. w/ $(d+1)$ unknowns, $(d+1)$ constraints.

full rank $\Rightarrow$ $\exists$ unique sol$^{\underline{n}}$.

one can check $K_{d,d}$ satisfies all C1 – C3

- To solve it,

  iterate $(d-k+1)\lambda p_{k-1} = k p_k$ for $2 \le k \le d$ (C3)

  $$\Rightarrow p_k = \frac{(d-1)!}{k!\,(d-k)!} \lambda^{k-1} p_1 \quad \dots\dots (1)$$

plug (1) into C2 $\Rightarrow$ $\dfrac{p_1}{d\lambda} = \dfrac{p_0}{(1+\lambda)^d}$ . . . . (2)

(1), (2) into C1 $\Rightarrow$ . . . $\blacksquare$.

Part II Polynomial methods.

- To use the info of roots of polyn. to solve some comb. problem.

§ Disc. Kakeya problem.

An example of using that low deg polyn. cannot have too many roots to show that certain comb. structure cannot be too small.

- We say a polyn. is a 0-polyn. if the coeff. of all its monomials are zero.
- We say a polyn. f vanish on a set A, or f is identically 0 on A, if $f(a)=0 \quad \forall a \in A$.

Note the diff: f polyn on $\mathbb{F}_p^n$, i.e. n-variate polyn in $\mathbb{F}_p[x_1, \dots, x_n]$

f could be identically 0 everywhere (on $\mathbb{F}_p^n$) but not 0-polyn.

e.g. $f(x) = x^p - x$. When $n = 1$.

<u>Fact</u>: • $\forall f$ non-zero deg-d polyn on $\mathbb{F}_p^n$

• $\forall a, z \in \mathbb{F}_p^n$ w./ $z \neq 0$, let $L = \{a + tz : t \in \mathbb{F}_p\}$ be the corresp. line. Then the restriction of f on L, denoted by $f_L(t)$, is a univariate polyn (of t) w./ deg $\leq d$ & leading coeff. $f_d(z)$, where $f_d(z)$ is the homogeneous ~~d~~ deg-d part of f.

<u>Pf</u>: $\forall$ monomials $\prod_{i=1}^n x_i^{r_i}$ has value at $a + tz$: $\prod_{i=1}^n \underline{(a_i + tz_i)}^{r_i}$

to get t^d term, need to choose tz_i term in each braket. $\Rightarrow t^d \prod_{i=1}^n z_i^{r_i}$ $f_d(z)$ ∎

Lem f non-zero, $\deg < p$, polyn on $\mathbb{F}_p^n$

$\Rightarrow$ f <u>cannot</u> vanish on the whole $\mathbb{F}_p^n$.

Pf: Induction on n.

- Base: $n=1$, if f vanishes on $\mathbb{F}_p$

  $\geq p$ roots $\Rightarrow$ $\deg f \geq p$ $\lightning$.

- General n. Supp. f vanishes on $\mathbb{F}_p^n$.

- Think of f as a polyn. of x_1 w/ coeff. in $\mathbb{F}_p[x_2, \dots, x_n]$

By polyn division alg.

$\Rightarrow \forall\, a \in \mathbb{F}_p$, $f(x) = P(x_1, \dots, x_n)(x_1 - a) + Q(x_2, \dots, x_n)$

- As $Q(x_2, \dots, x_n) = f(a, x_2, \dots, x_n)$ vanishes on $\mathbb{F}_p^{n-1}$

(IH) $\Rightarrow$ Q is 0-polyn.

$\Rightarrow$ $(x_1 - a)$ divides f $\forall\, a \in \mathbb{F}_p$

$\Rightarrow$ f has $\geq p$ roots $\Rightarrow$ $\deg \geq p$ $\lightning$ $\blacksquare$
(as a univar. poly. of x_1)