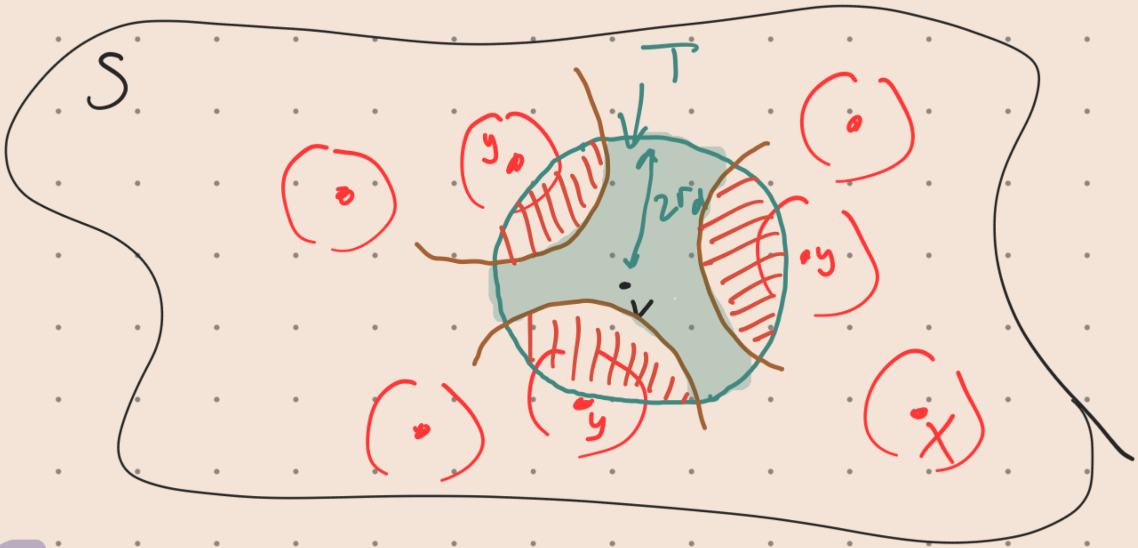
Lecture 6

Recall

$$T = T(X,v) = \left\{ x \in B_{2r_d}(v) \cap S : d(x,y) > 2r_d \,, \forall y \in X \cap B^c_{2r_d}(v) \right\}$$



Idea:

- Experiment $\begin{cases} 1. & X \sim_{HS(\lambda)} S \\ 2. & \text{Sample unif pt } v \sim S \end{cases}$

- bound $\alpha_S(\lambda)$ in two ways $\begin{cases} (1) & \text{Use free vol (if } v \in X) \\ (2) & \text{resample on } T \end{cases}$

- Let $t = \text{vol}(T)$

$$\alpha_S(\lambda) \geq \max \left\{ \overset{(1)}{\lambda \cdot \mathbb{E}_{X,v}\left(e^{-\lambda t}\right)}, \overset{(2)}{2^{-d}\, \mathbb{E}\left(\alpha_T(\lambda)\cdot t\right)} \right\}$$

- For (2) bound estimate

expectation of Poisson pt process

Lem1 $\forall \beta, \exists k_0 : k_0 \leq k \leq \lambda t \implies \alpha_T(\lambda)\cdot t \geq (1-\beta) p_k \cdot k$,

where $p_i = \mathbb{P}(\text{unif indep } i \text{ pts in } T \text{ dist} \geq 2r_d \text{ apart})$

Lem2. T meas. w/ vol $t \in [2^{d/2}, 2^d]$ & $u \sim T$ unif pt in T. $\Rightarrow$ $\mathbb{E}_u[\mathrm{vol}(B_{2r_\delta}(u) \cap T)] \le 2 \cdot 2^d (1 - t^{-2/d})^{d/2}$.

- Monotonicity of $\alpha_S(\lambda)$ w.r.t. λ $\Rightarrow$ $\lambda = (1/\sqrt{2} - \delta)^d$

NTS $\boxed{\alpha_S(\lambda) \ge (\log\sqrt{2} - \varepsilon)\, d \cdot 2^{-d}}$.

- $k = (\log\sqrt{2} - \varepsilon/2) \cdot d$

L set of pts whose corresp. T is small

- $L(X) = \{u \in S :\ t(X,u) \le k/\lambda\}$
- By (1), $\alpha_S(\lambda) \ge \lambda \cdot \mathbb{E}_{X,v}\left[e^{-\lambda t(X,v)}\right]$

$\forall v \in L$ $\lambda \cdot t(X,v) \le k$ $\Rightarrow \alpha_S(\lambda) \cdot \mathrm{vol}(S) \ge \lambda \mathbb{E}_X \int_{v \in S} e^{-\lambda t(X,v)} dv$ $\to v \in L$

$$\cdots \ge e^{\varepsilon d/2} \cdot 2^{-d}\, \mathbb{E}_X(\mathrm{vol}(L))$$

$\Rightarrow$ May assume $\mathbb{E}_X(\mathrm{vol}(L)) \le \mathrm{vol}(S) \cdot e^{-\varepsilon d/4}$, o.w. ☺

- Markov ineq $\Rightarrow$ $\boxed{\mathbb{P}_X\left(\mathrm{vol}(L) \ge \mathrm{vol}(S) \cdot e^{-\varepsilon d/6}\right) \le e^{-\varepsilon d/12} \quad (\heartsuit)}$

In words, for typical outcome X, t is 'relatively large' except for a very small of pts in S.

- Take any X w./ $\mathrm{vol}(L) \le \mathrm{vol}(S) \cdot e^{-\varepsilon d/6}$

$$\forall\ v \in S \setminus L, \quad \underbrace{t = t(X,v) \ge k/\lambda}_{k \le \lambda t} \ge (\sqrt{2} + \delta/3)^d$$

Also $t \le 2^d$ as $T \in B_{2r_d}(\cdot)$

- So Lem 1 applies $\Rightarrow$ $\forall\, v \in S \setminus L$,

$\mathbb{P}(v_1, \dots, v_k \sim T,\ d(v_i, v_j) \geq 2r_d)$

$$\alpha_T(\lambda) \cdot t \geq (1-\beta)\, p_k \cdot k$$

<u>Claim</u> $p_k \geq 1-\delta$

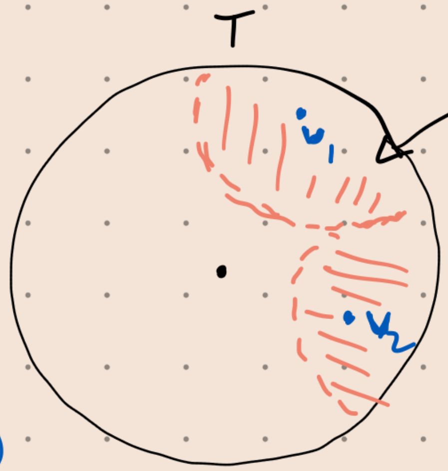
<u>Pf</u>: <u>Idea</u>

greedily pick k pts

each pt takes up

negligible (exp. in d small)

portion of T. (by Lem 2)



- each part $e^{-\Omega(d)}$ small

- only need $k = \Theta(\log d)$

- Consider $g(t) := (f(\tau))^{-d}$, where $\begin{cases} \tau := t^{1/d} \\ f(\tau) = \dfrac{\tau}{2\sqrt{1-\tau^{-2}}} \end{cases}$

- Obs. $f(\sqrt{2}) = 1$, f strictly $\nearrow$ on $[\sqrt{2}, 2]$

Since $f'(x) = \cdots = \dfrac{x^2 - 2}{2\sqrt{1-\frac{1}{x^2}}\,(x^2-1)} > 0$.

$v \notin L$

Recall $t \geq (\sqrt{2} + \delta/3)^d \Rightarrow f(t^{1/d}) \geq f(\sqrt{2} + \delta/3) > 1$.

$\Rightarrow$ i.e. $g(t)$ is exp. small in d.

- Lem 2 $\Rightarrow$ $2g(t)$ upper bounds expected fraction of meas. of $T \cap$ ball of radius $2r_d$ centred at a unif pt of T.

• Let $x_1, \cdots, x_k \in T$ indep unif pt.

Call x_i bad if
$$\begin{cases} \bullet\ \underbrace{vol(B_{2r_d}(x_i) \cap T) \geq t/d^3}_{E_1} \\ \text{OR} \\ \bullet\ \underbrace{\text{within dist } 2r_d \text{ from } x_1, \cdots, x_{i-1}}_{E_2} \end{cases}$$

• Suffices to show $\mathbb{P}(\exists \geq \text{one bad } x_i) = o(1)$

$\Leftarrow$ For each i, $\mathbb{P}(x_i \text{ is the } 1^{st} \text{ bad } x) = o(1/k)$

This prob. $\leq \underbrace{\Pr(E_1^c)}_{= e^{-\Omega(d)}} + \underbrace{\Pr(E_2^c \mid E_1^c)}_{\text{Lem 2 by Markov}} \leq \frac{i-1}{d^3}$ ▨

• By Lem 1, $\underline{\forall\, v \in S \setminus L}$,

$$\alpha_T(\lambda) \cdot t \geq (1-\beta) p_k \cdot k \overset{\text{Claim}}{\geq} (1-2\beta) k$$

• By ② $\quad 2^d \cdot \alpha_S(\lambda) \geq \mathbb{E}_{x,v}\left(\alpha_T(\lambda) \cdot t\right)$

$$= \frac{1}{vol(S)} \mathbb{E}_X \left[\int_{v \in S} \alpha_T(\lambda) \cdot t \, dv \right]$$

$$\geq \frac{1}{vol(S)} \mathbb{E}_X \left[\int_{v \in S \setminus L} \alpha_T(\lambda) \cdot t \, dv \right]$$

only take X s.t. $vol(L) \leq vol(S) \cdot e^{-\varepsilon d/6}$ (B)

$$\geq \left(1 - e^{-\varepsilon d/12}\right)\left(1 - e^{-\varepsilon d/6}\right)(1-2\beta) k$$

$$= (1-o(1))\left(\ln\sqrt{2} - \tfrac{\varepsilon}{2}\right) d$$

$$\Rightarrow \quad \alpha_S(\lambda) \geq (\ln\sqrt{2} - \varepsilon)\, d \cdot 2^{-d}$$

<u>Def</u>: $i(G)$ = # indep sets in G

bip G reduce it to bip G

<u>Thm1</u> [Kahn - Zhao]

$\forall$ n-vx d-reg G, $\quad i(G) \leq i(K_{d,d})^{n/2d}$

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$n/2d$ many

Extension

Thm2 (Davies - Jessen - Perkins - Roberts)

$\forall$ d-reg G $\forall \lambda > 0$

$$\bar{\alpha}_G(\lambda) \leq \bar{\alpha}_{K_{d,d}}(\lambda) = \frac{\lambda(1+\lambda)^{d-1}}{2(1+\lambda)^d - 1}.$$