Lecture 5

Recall

- Hard-sphere model over bdd. meas. $S \subseteq \mathbb{R}^d$ w/ fugacity $\lambda > 0$.

  Partition funct. $Z_S(\lambda) = \sum_{k \geq 0} \lambda^k \hat{Z}_S(k)$ → uniform over slices; → k^{th} slice weighted by λ^k.

  $$\hat{Z}_S(k) = \frac{1}{k!} \int_{S^k} \mathbb{1}_{D(x_1, \cdots, x_k)} \, dx_1 \cdots dx_k = \text{vol. size-}k \text{ packings.}$$

  ← event : $d(x_i, x_j) > 2r_d$

  - $\hat{Z}_S(0) = 1$
  - $X \sim_{HS(\lambda)} S \Rightarrow \mathbb{P}(|X| = k) = \dfrac{\lambda^k \hat{Z}_S(k)}{Z_S(\lambda)}$

- Expected packing density

  $$\alpha_S(\lambda) = \frac{\mathbb{E}_{S,\lambda} |X|}{\text{vol}(S)}$$

<u>Thm</u> [Gil-Fernandéz, Kim, Liu, Pikhurko 21+]

$\forall \varepsilon > 0 \ \exists \delta > 0, d_0$ s.t. $\forall d > d_0$, $\forall \lambda \geq \left(\frac{1}{\sqrt{2}} - \delta\right)^d$

$$\Rightarrow \alpha_S(\lambda) \geq \left(\log \sqrt{2} - \varepsilon\right) \cdot d \cdot 2^{-d}.$$

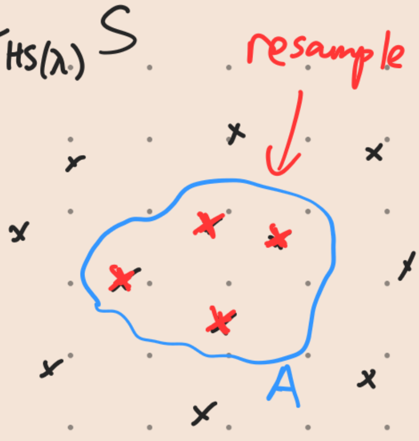
<u>Prop 1</u>. $\alpha_S(\lambda) = \dfrac{\lambda}{\text{vol}(S)} \left(\log Z_S(\lambda)\right)'$.

<u>Lem 2</u>. $\alpha_S(\lambda)$ is monotone increasing wrt λ.

Spatial Markov property:

- Sample $X \sim_{HS(\lambda)} S$
- $\forall$ meas. A, resample on externally uncovered part of A

  $\Rightarrow$ result in the same dist. as X.



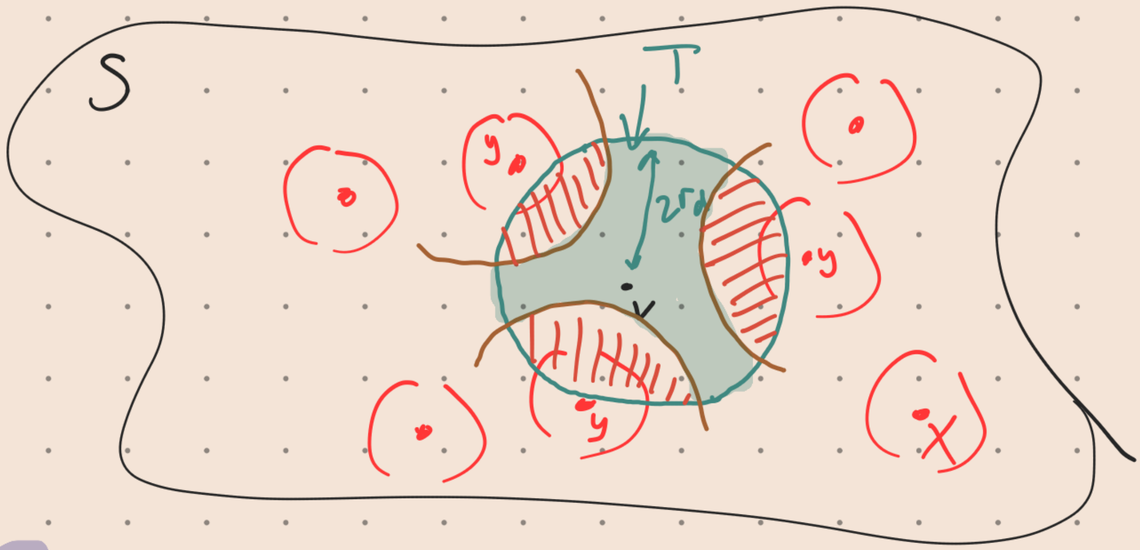
- Expected free volume

$$FV_S = \frac{1}{vol(S)} \int_S \mathbb{P}(\, d(x_0, X) > 2r_d)\, dx_0$$

prob. that a unif random pt can be added

- Externally uncovered part: given a set $X \subseteq S$, a pt v.

$$T = T(X,v) = \{ x \in B_{2r_d}(v) \cap S : d(x,y) > 2r_d \,, \forall y \in X \cap B^c_{2r_d}(v) \}$$



Idea:

- Experiment $\begin{cases} 1. & X \sim_{HS(\lambda)} S \\ 2. & \text{Sample unif pt } v \sim S \end{cases}$

- bound $\alpha_S(\lambda)$ in two ways $\begin{cases} (1) & \text{Use free vol (if } v \in X) \\ (2). & \text{resample on } T \end{cases}$

(1). Lem 3. (i) $\alpha_S(\lambda) = \lambda \cdot FV_S$

(ii) $\alpha_S(\lambda) = \lambda \cdot \mathbb{E}\left[\frac{1}{Z_T(\lambda)} \right]$

(iii) $\alpha_S(\lambda) \geq \lambda \cdot \mathbb{E}_{X,v}\left(e^{-\lambda \cdot vol(T)} \right)$

pf: (ii) $\alpha_S(\lambda) = \lambda \cdot FV_S$

$$= \lambda \cdot \frac{1}{vol(S)} \int_S \mathbb{P}(d(v,X) > 2r_d)\, dv = \lambda \cdot \mathbb{E}\left[\mathbb{1}_{\{T \cap X = \phi\}} \right]$$

$$= \lambda \, \mathbb{E}\left[\frac{1}{Z_T(\lambda)}\right]$$

(iii) Recall $\quad Z_S(\lambda) = \sum_{k \geq 0} \lambda^k \hat{Z}_S(k) = \sum_{k \geq 0} \frac{\lambda^k}{k!} \int_{S^k} \mathbb{1}_{D(x_1, \dots, x_k)} \, dx_1 \dots dx_k$

$$\leq \sum_{k \geq 0} \frac{\lambda^k}{k!} \mathrm{vol}(S)^k = e^{\lambda \cdot \mathrm{vol}(S)}$$

By (ii), $\quad \alpha_S(\lambda) = \lambda \cdot \mathbb{E}\left[\frac{1}{Z_T(\lambda)}\right] \overset{(iii)}{\geq} \lambda \cdot \mathbb{E}_{X,v} \, e^{-\lambda \cdot \mathrm{vol}(T)}$ $\blacksquare$

② Lem 4. $\quad \alpha_S(\lambda) \geq 2^{-d} \cdot \mathbb{E}\left[\alpha_T(\lambda) \cdot \mathrm{vol}(T)\right]$

Pf: $\quad \alpha_S(\lambda) = \frac{1}{\mathrm{vol}(S)} \cdot \mathbb{E}|X|$

$$\geq 2^{-d} \cdot \mathbb{E}\left|X \cap B_{2r_d}(v)\right|$$

as $\mathrm{vol}(S \cap B_{2r_d}(v)) \leq 2^d \quad \forall \, v \in S$

Spatial Markov $\quad = 2^{-d} \, \mathbb{E}\left[\alpha_T(\lambda) \cdot \mathrm{vol}(T)\right]$

So we have, writing $t = \mathrm{vol}(T)$

$$\alpha_S(\lambda) \geq \max\left\{ \lambda \cdot \mathbb{E}_{X,v}\left(e^{-\lambda t}\right), \; 2^{-d} \, \mathbb{E}\left(\alpha_T(\lambda) \cdot t\right) \right\}$$

large when t small $\qquad\qquad$ large when t big

For the 2^{nd} bound, we use the following

Lem5. $\forall \beta > 0$, $\exists k_0$ s.t. $\forall k \geq k_0$, $\forall \lambda, t, d > 0$,

if meas. $T \subseteq \mathbb{R}^d$ is of vol. t and $k \in \mathbb{N}$: $k_0 \leq k \leq \lambda t$

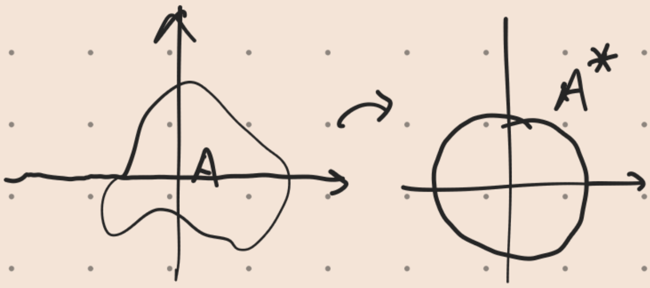
$\Rightarrow \alpha_T(\lambda) \cdot \mathrm{vol}(T) \geq (1-\beta)\, p_k \cdot k$, where

$p_i = \mathbb{P}$(unif indep i pts in T are at pairwise dist. $\geq 2r_d$).

For the 1st bound

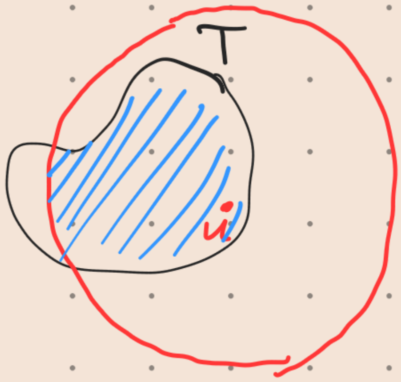
Def. A meas. set, symmetric rearrangement

$$A^* = B_{\mathrm{vol}(A)^{1/d} \cdot r_d}(0)$$



For meas. $T \subseteq \mathbb{R}^d$,

define $f(T) = \int_T \mathrm{vol}\left(B_{2r_d}(u) \cap T\right) du$



Lem6. $\forall$ bdd meas. $T \subseteq \mathbb{R}^d$,

$$f(T) \leq f(T^*).$$

- By Riesz's rearrangement inequality.

Lem7. T meas. in $\mathbb{R}^d$ of vol. $t \in [2^{d/2}, 2^d]$.

Let $u \sim T$ a unif random pt in T,

$$\Rightarrow \quad \mathbb{E}_u\left[\mathrm{vol}\left(B_{2r_d}(u) \cap T\right)\right] \leq 2 \cdot 2^d \left(1 - t^{-2/d}\right)^{d/2}.$$

<u>Pf</u>: • $\mathbb{E}_u\left[\mathrm{vol}\left(B_{2r_d}(u) \cap T\right)\right] = \dfrac{f(T)}{\mathrm{vol}(T)}$

By <u>Lem 6</u>, may assume T is a ball of radius

$\rho = t^{1/d} \cdot r_d$, $\quad T = B_\rho(0)$

$$\mathbb{E} \cdots = \frac{1}{t} \int_T \left(\int_T \mathbb{1}_{\{d(u,v) \leq 2r_d\}} \, dv \right) du$$

$$= \frac{2}{t} \int_T \int_T \mathbb{1}_{\{d(u,v) \leq 2r_d\}} \cdot \underline{\mathbb{1}_{\{\|v\| \leq \|u\|\}}} \, dv \, du$$

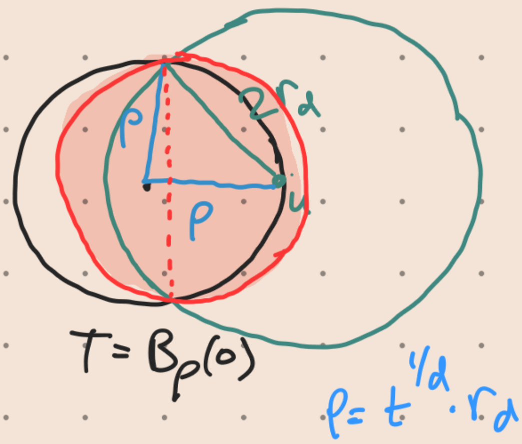
$$= 2 \cdot \max_{u \in B_\rho(0)} \int_T \mathbb{1}_{\{d(u,v) \leq 2r_d\}} \cdot \mathbb{1}_{\{\|v\| \leq \|u\|\}} \, dv$$

May assume $\rho \geq \sqrt{2}\, r_d$,

o.w. $t \leq 2^{d/2}$.

radius of the red ball

$= 2\sqrt{1 - t^{-2/d}} \cdot r_d$



$$\leq \max\left\{ 2^{d/2}, \; 2 \cdot \max_{\sqrt{2} \leq x \leq t^{1/d}} \left(2\sqrt{1 - x^{-2}}\right)^d \right\}$$

$$= 2 \cdot \left(2\sqrt{1 - t^{-2/d}}\right)^d. \qquad \blacksquare$$

Pf (Main thm) Given $\varepsilon > 0$, choose $\beta \gg \delta > 0$.

Let $d \to \infty$.

- $\alpha_S(\lambda)$ is non-decreasing in λ, enough to consider $\lambda = (1/\sqrt{2} - \delta)^d$.

- Let S be a large ball in $\mathbb{R}^d$,

NTS : $\boxed{\alpha_S(\lambda) \geq (\log \sqrt{2} - \varepsilon) \cdot d \cdot 2^{-d}.}$ ☺

Experiment $\begin{cases} X \sim_{HS(\lambda)} S \\ v \sim S \end{cases} \Rightarrow T = T(X, v)$

- Define $k = (\log \sqrt{2} - \varepsilon/2) \cdot d$.

For $X \in S$, let

$$L = L(X) = \{u \in S : t(X,u) \leq k/\lambda\}$$

$\forall v \in L, \quad \lambda \cdot t(X,v) \leq k$

- Using ①

$$\Rightarrow \alpha_S(\lambda) \geq \lambda \cdot \mathbb{E}_X \mathbb{E}_v e^{-\lambda t(X,v)}$$

$$\mathrm{vol}(S) \cdot \alpha_S(\lambda) \geq \lambda \mathbb{E}_X \left[\int_{v \in S} e^{-\lambda t(X,v)} dv \right]$$

$$\geq \lambda \mathbb{E}_X \left[\int_{v \in L} e^{-\lambda t(X,v)} dv \right]$$

Calculation ......

$$\geq e^{\varepsilon d/3} \cdot 2^{-d} \cdot \mathbb{E}_X [\mathrm{vol}(L)].$$

- $\Rightarrow$ May assume $\mathbb{E}_X[\mathrm{vol}(L)] \leq \mathrm{vol}(S) \cdot e^{-\varepsilon d/4}$

  o.w. ☺ holds.

  - Markov's ineq.

    $$\Rightarrow \mathbb{P}_X\left(\mathrm{vol}(L) \geq \mathrm{vol}(S) \cdot e^{-\varepsilon d/6}\right) \leq e^{-\varepsilon d/12}.$$

That is, for a typical outcome X,

t is relatively large.