Lecture 31

Recall

Def [Bounded differences condition]

A funct. $f : \Omega_1 \times \cdots \times \Omega_n \to \mathbb{R}$ has the bounded differences condition w./ parameter $(c_1, \ldots, c_n) \in \mathbb{R}^n$ if $\forall i \in [n]$ $\forall x_i, x_i' \in \Omega_i$,

$$\left| f(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n) - f(x_1, \ldots, x_{i-1}, x_i', x_{i+1}, \ldots, x_n) \right| \leq c_i$$

Lem 1 Suppose $g : \{0,1\}^n \to \mathbb{R}$ has bounded diff. cond. w./ parameter $(c_1, \ldots, c_n)$ and $\xi \in \{0,1\}^n$ is a random variable uniformly distributed in $\binom{[n]}{k}$. Then

$$\mathbb{P}\left(|g(\xi) - \mathbb{E} g(\xi)| \geq t \right) \leq 2 e^{-\frac{t^2}{8 \sum c_i^2}} \quad \text{for all } t \geq 0.$$

Lem 2 Suppose $f : \{0, 1, \ldots, q-1\}^n \to \mathbb{R}$ has bounded diff. cond. w./ parameters $(c_1, \ldots, c_n)$ and η is drawn uniformly at random from $\{0, 1, \ldots, q-1\}^n$ subject to $wt(\eta) = k$.

Then
$$\mathbb{P}\left(|f(\eta) - \mathbb{E} f(\eta)| \geq t \right) \leq 2 e^{-\frac{t^2}{68 \sum c_i^2}} \quad \text{for all } t \geq 0.$$

We will use Lem 1 & the following standard fact about subgaussian r.v. to prove Lem 2.

Lem 3. Let X be a r.v. w/ mean zero. Then the following properties are equivalent.

(i) The tails of X satisfy

$$\mathbb{P}(|X| \geq t) \leq 2e^{-t^2/K_1} \qquad \forall t \geq 0$$

(ii) The moment generating funct. of X satisfies

$$\mathbb{E} e^{\lambda X} \leq e^{K_2 \cdot \lambda^2} \qquad \forall \lambda \geq 0$$

In particular, for (i) $\Rightarrow$ (ii) we can take $K_2 = 2K_1$

(ii) $\Rightarrow$ (i) . - - - - $K_1 = 4K_2$.

Pf (Lem 2). • Let $\xi \in \{0,1\}^n$ be a r.v. uniformly distributed in $\binom{[n]}{k}$

• Let u be drawn uniformly from $[q-1]^n = \{1, \dots, q-1\}^n$, independent of ξ.

$\Longrightarrow$ the distribution of η coincides w/ the distribution of $u * \xi = (u_1\xi_1, \dots, u_n\xi_n)$. $\sum c_i^2$

• By Lem 3, it suffices to show that

$$\mathbb{E}_u \mathbb{E}_\xi \, e^{\lambda\left(f(u*\xi) - \mathbb{E}_{u,\xi} f(u*\xi)\right)} \leq e^{17\|c\|^2 \lambda^2}$$

- Fix an instance of u. Note that the funct.

$f(u * \cdot)$ has the bounded diff. cond. w/

param. $c = (c_1, \dots, c_n)$.

$\dots \xi_i \dots \dots \xi_i' \dots$

$|f(u*\xi) - f(u*\xi')| \le c_i$

- Then by Lem 1 w/ $f(u* \cdot)$ playing the role of $g(\cdot)$ and Lem 3, we get

$$\mathbb{E}_\xi \, e^{\lambda(f(u*\xi) - \mathbb{E} f(u*\xi))} \le e^{16\|c\|^2\lambda^2}. \quad (*)$$

Thus, $\mathbb{E}_u \mathbb{E}_\xi \, e^{\lambda(f(u*\xi) - \mathbb{E}_{u,\xi} f(u*\xi))}$

$$= e^{-\lambda \mathbb{E}_{u,\xi} f(u*\xi)} \cdot \mathbb{E}_u \, e^{\lambda \mathbb{E}_\xi f(u*\xi)} \, \mathbb{E}_\xi \, e^{\lambda(f(u*\xi) - \mathbb{E}_\xi f(u*\xi))}$$

$$\le e^{16\|c\|^2\lambda^2} \, \mathbb{E}_u e^{\lambda(\mathbb{E}_\xi f(u*\xi) - \mathbb{E}_{u,\xi} f(u*\xi))}$$

- As $g(\cdot) := \mathbb{E}_\xi f(\cdot * \xi)$ has bounded diff. cond. w/ param. c, by McDiarmid's ineq.

$$\mathbb{P}\left(|g(u) - \mathbb{E}_u g(u)| \ge t\right) \le 2e^{-2t^2/\|c\|^2}.$$

By Lem 3 $\Rightarrow$ $\mathbb{E}_u \, e^{\lambda(g(u) - \mathbb{E} g(u))} \le e^{\|c\|^2\lambda^2}$

Application

We shall see a geometric application of the large deviation ineq over a slice.

The question to consider is that given a metric space (X, d), how can we bound the volume of intersection of two balls.

We will prove a result providing natural sufficient cond. on the metric space (X,d) guaranteeing exponential decay on intersection volume.

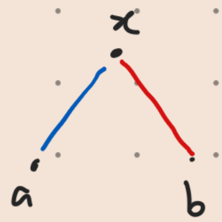
<u>Def</u>: A metric space (X,d) has exponential growth at radius r w/ rate c if $\forall a \in X$ and $\forall t < r$,

$$\frac{\mathrm{vol}(B(a, r-t))}{\mathrm{vol}(B(a,r))} \leq 2e^{-ct}.$$

ball centered at a w/ radius $r-t$.

<u>Def</u>. For $a, b \in X$, let $l_{a,b} : X \to \mathbb{R}$ be

$$l_{a,b}(x) = d(x,b) - d(x,a).$$

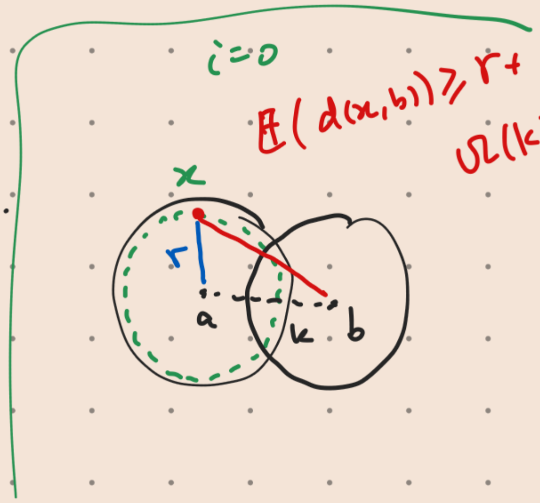


Given $r, k \in \mathbb{N}$ and $\alpha > 0$, we say that the metric space (X, d) is (r,k)-dispersed w/ constant α if $\forall\, a, b \in X$ w/ $d(a,b) = k$

and any $0 \le i \le \alpha k$,

$$\mathop{\mathbb{E}}_{x \sim \underline{S(a,\, r-i)}} \left[l_{a,b}(x) \right] \geq 2 \alpha k .$$

all pts of distance $r-i$ from a.



- Recall a real-valued r.v. X is K-subgaussian if

$$\forall\, t \geq 0 \qquad \mathbb{P}(|X| \geq t) \le 2 e^{-t^2/K} .$$

<u>Thm</u> (Kim, Lin, Tran) Let (X, d) be a finite metric space w/ d taking values in $\mathbb{N} \cup \{0\}$ and let $k, r \in \mathbb{N}$. Suppose

(A1) (X, d) has exponential growth at radius r w/ rate $c > 0$.

(A2) (X, d) is (r,k)-dispersed w/ const. $\alpha > 0$.

(A3) $\forall\, a, b \in X$ w/ $d(a,b) = k$ and $\forall\, 0 \le i \le \alpha k$

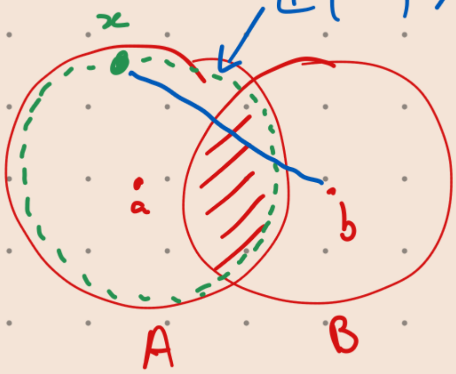
$\underline{l_{a,b}(x) - \mathbb{E}\, l_{a,b}(x)}$ is K-subgaussian, where x is drawn uniformly from $S(a, r-i)$.

$\Longrightarrow$ Then $\forall\, a,b \in X$ w./ $d(a,b)=k$,

$$\frac{\mathrm{vol}\big(B(a,r)\cap B(b,r)\big)}{\mathrm{vol}\big(B(a,r)\big)} = 2e^{-\Omega_{c,\alpha}\left(k+\frac{k^2}{K}\right)}$$

Idea

$\mathbb{E}(\ \ ) \geq r+\Omega(k)$



$x \sim A$

$$\frac{\mathrm{vol}(A\cap B)}{\mathrm{vol}(A)} \quad \text{small}$$

$$\|$$

$$P(x\in A\cap B)$$

Pf (Thm) Let $T = B(a,r)\cap B(b,r)$ and $\eta \sim B(a,r)$. Then

$$\frac{\mathrm{vol}\big(B(a,r)\cap B(b,r)\big)}{\mathrm{vol}\big(B(a,r)\big)} = \mathbb{P}(\eta\in T) = \mathbb{P}\big(d(\eta,b)\leq r\big)$$

By (A1) $\Rightarrow$ $\mathbb{P}\big(d(\eta,a)\leq r-\alpha k\big) \leq 2e^{-\Omega(k)}$.

Thus,
$$\mathbb{P}(\eta\in T) \leq \mathbb{P}\big(\eta\in T \,\big|\, d(\eta,a) > r-\alpha k\big)\cdot \mathbb{P}\big(d(\eta,a)>r-\alpha k\big) + \mathbb{P}\big(d(\eta,a)\leq r-\alpha k\big)$$

$$\leq \sum_{i=0}^{\alpha k} \mathbb{P}\big(d(\eta,b)\leq r \,\big|\, d(\eta,a)=r-i\big)\,\mathbb{P}\big(d(\eta,a)=r-i\big)$$

$$+ 2e^{-\Omega(k)}$$

$$\leq \max_{0 \leq i \leq \alpha k} \mathbb{P}\left(d(\eta,b) \leq r \,\middle|\, \underbrace{d(\eta,a) = r-i}\right) + 2e^{-\Omega(k)}.$$

- Fix an $i \in [0, \alpha k]$. and let $x \sim S(a, r-i)$

Note that, conditioning on $d(\eta, a) = r-i$, η and x are identically distributed.

$$\Rightarrow \mathbb{P}\left(d(\eta,b) \leq r \,\middle|\, d(\eta,a) = r-i\right) = \mathbb{P}\left(d(\eta,b) - d(\eta,a) \leq i \,\middle|\, d(\eta,a) = r-i\right)$$

$$= \mathbb{P}\left(d(x,b) - d(x,a) \leq i\right)$$

$$= \mathbb{P}\left(l_{a,b}(x) \leq i\right)$$

- (A2) $\Rightarrow$ $\mathbb{E}\, l_{a,b}(x) \geq 2\alpha k$.

$$\Rightarrow \underline{i - \mathbb{E}\, l_{a,b}(x) \leq i - 2\alpha k \leq -\alpha k}$$

by (A3)

Thus, since $l_{a,b}(x) - \mathbb{E}\, l_{a,b}(x)$ is K-subgaussian

$$\Rightarrow \mathbb{P}\left(l_{a,b}(x) \leq i\right) = \mathbb{P}\left(\underline{l_{a,b}(x) - \mathbb{E}\, l_{a,b}(x) \leq i - \mathbb{E}\, l_{a,b}(x)}\right)$$

$$\leq \mathbb{P}\left(\ell_{a,b}(x) - \mathbb{E}\ell_{a,b}(x) \leq -\alpha k \right)$$

$$\leq 2e^{-\Omega(k^2/K)} \qquad \blacksquare$$