Lecture 30

- Simple scenario: $S_n = X_1 + \cdots + X_n$, X_i, iid r.v.

$$-n \le S_n \le n \qquad X_i = \begin{cases} 1 & 1/2 \\ -1 & 1/2 \end{cases}$$

But typically value of S_n is sharply concentrated in a small window (width $O(\sqrt{n})$)

<u>Thm</u> [Chernoff] $X_1, \dots, X_n$ iid Rademacher r.v.

$$S_n = X_1 + \cdots + X_n$$

$$\Rightarrow \forall a > 0, \quad \mathbb{P}(|S_n| > a) < 2e^{-a^2/2n}$$

<u>Idea</u>: apply Markov to exponential moment.

<u>Pf</u>: • By symm, suffices to show

$$\mathbb{P}(S_n > a) < e^{-a^2/2n}.$$

- Consider the exponential moment of each X_i.

$\underline{\forall \lambda > 0}$: $$\mathbb{E}(e^{\lambda X_i}) = \frac{e^{\lambda} + e^{-\lambda}}{2} = \cosh(\lambda) \le e^{\lambda^2/2}$$

by comparing Taylor series

- indep. of X_i's $\Rightarrow$ $$\mathbb{E}(e^{\lambda S_n}) = \mathbb{E}\left(e^{\lambda \sum_{i=1}^n X_i}\right)$$

$$= \mathbb{E}\left(\prod e^{\lambda X_i}\right)$$

$$\overset{\text{indep}}{=} \prod_{i=1}^{n} \mathbb{E} e^{\lambda X_i} \leq e^{\lambda^2 n/2}$$

- $$\mathbb{P}(S_n > a) = \mathbb{P}\left(e^{\lambda S_n} > e^{\lambda a}\right)$$

$$\overset{\text{Markov}}{<} \frac{\mathbb{E}\left(e^{\lambda S_n}\right)}{e^{\lambda a}} \leq e^{\lambda^2 n/2 - \lambda a}$$

<u>RHS</u> optimised when $\lambda = a/n$.

<u>Many extensions:</u>

- X_i : bounded r.v.
- sum $S_n = \sum X_i$

$\rightsquigarrow$ $f(X_1, \dots, X_n)$ w/ Lipschitz condition.

<u>Thm</u> [Hoeffding's ineq]

$X_1, \dots, X_n$ indep r.v., $X_i \in [a_i, b_i]$

Let $S_n = \sum X_i$, $\sigma^2 = \sum_{i=1}^{n} |b_i - a_i|^2$

$$\Rightarrow \mathbb{P}\left(|S_n - \mathbb{E}S_n| \geq \underbrace{\lambda \cdot \sigma}_{a} \right) \leq C \cdot e^{-c\lambda^2}.$$

$\Updownarrow$ Same

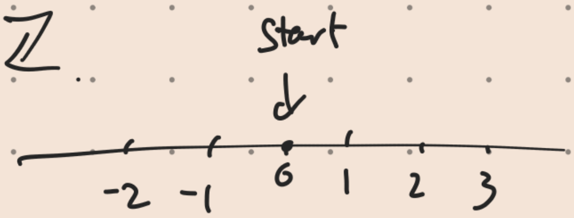
<u>Rmk</u>: $\mathbb{P}\left(|S_n - \mathbb{E}S_n| > a \right) \leq C \cdot e^{-\frac{ca^2}{\sum_{i=1} |b_i - a_i|^2}}$

<u>Def</u>: A seq of r.v. $X_1, \dots, X_n, \dots$ is a martingale if

$$\begin{cases} \cdot\ \mathbb{E}|X_n| < \infty \\ \cdot\ \mathbb{E}\left(X_{n+1} \mid X_n, \dots, X_1 \right) = X_n \end{cases}$$

<u>Example</u>: 1) random walk on $\mathbb{Z}$.

X_n: position at time n



2) gambler's fortune

X_n: total fortune at time n.

<u>Thm</u> [Azuma-Hoeffding's ineq]

$X_0, X_1, \dots, X_n$ martingale w/ $|X_i - X_{i-1}| \leq c_i$

$$\Rightarrow \forall a > 0, \quad \mathbb{P}\left(|X_n - X_0| \geq a \right) \leq 2e^{-\frac{2a^2}{\sum c_i^2}}$$

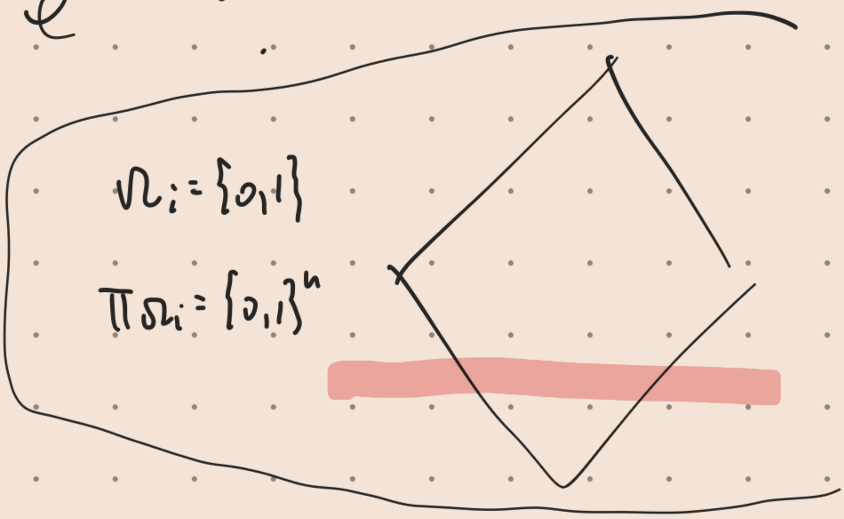
Using A-H $\Rightarrow$ large deviation for Lipschitz funct.

Def: (bounded difference) A function $f : \Omega_1 \times \cdots \times \Omega_n \to \mathbb{R}$ has bounded difference w/ parameter $(c_1, \ldots, c_n) \in \mathbb{R}^n$ if $\forall\, i \in [n]$ and $\forall\, x_i, x_i' \in \Omega_i$

$$\left| f(x_1, \cdots, x_{i-1}, x_i, x_{i+1}, \cdots, x_n) - f(\cdots\cdots, x_i', \cdots\cdots) \right| \le c_i .$$

Thm [McDiarmid's ineq]
Let $X_1, \cdots, X_n$ be r.v. w/ X_i taking values in Ω_i and let $f : \Omega_1 \times \cdots \times \Omega_n \to \mathbb{R}$ w/ bounded difference $(c_1, \ldots, c_n)$.
$\forall\, a > 0$

$$\Rightarrow \mathbb{P}\left(\left| f(X_1, \cdots, X_n) - \mathbb{E} f(X_1, \cdots, X_n) \right| \geq a \right) \le 2 e^{-\frac{2a^2}{\sum c_i^2}} .$$



We shall see some variation. A common prod. space is $\Omega_i = \{0,1\} \Rightarrow \{0,1\}^n$. The following is a large deviation ineq for Lipschitz funct. on a slice of Boolean cube.

<u>Lemma</u> [Kwan, Sudakov, Tran]

Suppose $g : \{0,1\}^n \longrightarrow \mathbb{R}$ satisfies the bounded difference cond. w/ parameter $(c_1, \dots, c_n)$ and $\xi \in \{0,1\}^n$ is a r.v. uniformly distributed in $\binom{[n]}{k}$. Then $\forall t > 0$.

$$\mathbb{P}\left(|g(\xi) - \mathbb{E}\, g(\xi)| \geq t \right) \leq 2 e^{-\frac{t^2}{8 \sum c_i^2}}.$$

<u>Pf</u>: • Assume w.l.o.g. that $c_1 \geq c_2 \geq \cdots \geq c_n$.

• Define $Z_i = \mathbb{E}\left[g(\xi) \,\middle|\, \xi_1, \dots, \xi_i \right]$

so $Z_0 = \mathbb{E}\, g(\xi)$

$Z_n = g(\xi)$

and $Z_0, \dots, Z_n$ is a martingale.

Let $\mathcal{L}(x_1, \dots, x_i)$ be the conditional distribution of ξ given $\xi_1 = x_1, \dots, \xi_i = x_i$.

<u>Claim</u>: $\left| \mathbb{E}\left(g(\mathcal{L}(x_1, \dots, x_{i-1}, 0))\right) - \mathbb{E}\left(g(\mathcal{L}(x_1, \dots, x_{i-1}, 1))\right) \right|$

$\leq 2c_i$ for any feasible choices of $x_1, \dots, x_{i-1} \in \{0,1\}$.

The claim implies that

$$|Z_i - Z_{i-1}| \leq 2c_i$$

and the conclusion follows from Azuma-Hoeffding.

(Pf of Claim) If $\xi \sim \mathcal{L}(x_1, \dots, x_{i-1}, 0 \to 1)$,

- change ξ_i to 1
- randomly choose one of the ones among $\xi_{i+1}, \dots, \xi_n$ and change it to 0.

obtain $\Rightarrow$ the distribution $\mathcal{L}(x_1, \dots, x_{i-1}, 1)$

This provides a coupling between $\mathcal{L}(x_1, \dots, x_{i-1}, 0)$ and $\mathcal{L}(x_1, \dots, x_{i-1}, 1)$ that differs in only two coordinates i and $j > i$,

as $c_j \leq c_i \Rightarrow$ total change $\leq 2c_i$. ▨

Lemma (Kim, Liu, Tran)

Suppose $f : \{0, 1, \dots, q-1\}^n \longrightarrow \mathbb{R}$ satisfies the bounded difference cond. w/ parameter $(c_1, \dots, c_n)$

and η is drawn uniformly at random from $\{0, 1, \cdots, q-1\}^n$ subject to $wt(\eta) = k$.

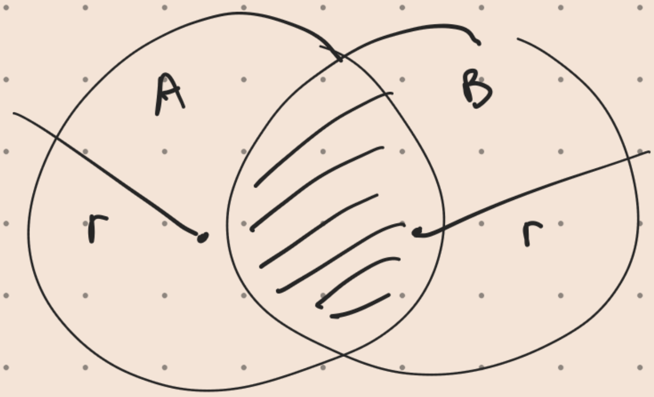
$wt(\eta) =$ # non-zero coordinates

$$\Rightarrow \mathbb{P}\left(|f(\eta) - \mathbb{E}f(\eta)| \geq t \right) \leq 2 e^{-\frac{t^2}{68 \sum c_i^2}} \quad \forall\, t \geq 0.$$

— Next time —

- pf of this lem using coupling.
- application: geometric lemma

$\{1, \cdots, q\}^n$



$\frac{vol(A \cap B)}{vol(A)}$ decays exponentially.