Lecture 3

Recall Sampling indep. sets in G

Hard-core model $\Pr(I) \propto \lambda^{|I|}$, λ fugacity

$$\Pr(I) = \frac{\lambda^{|I|}}{\underbrace{\sum_{J \in \mathcal{I}(G)} \lambda^{|J|}}_{\text{partition function}}}$$

Rmk: Among all distributions over $\mathcal{I}(G)$ with given mean size, the Hard-core dist. has the highest entropy.

Thm Δ-free, n-vx, max-deg $\leq d \Rightarrow \bar{\alpha}(G) \geq (1+o(1)) \frac{\log d}{d} \cdot n$

Conj (Davies-Jenssen-Perkins-Roberts)

$\exists\, c > 0$, s.t. $\forall$ Δ-free G,

$$\frac{\alpha(G)}{\bar{\alpha}(G)} \geq 1 + c.$$

Def: occupancy fraction

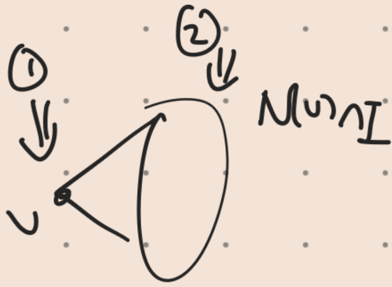
$$\frac{\bar{\alpha}_G(\lambda)}{n}$$

Thm $\forall \lambda > 0$, $\forall$ n-vx Δ-free G w/ $\Delta(G) \leq d$.

$$\frac{\bar{\alpha}_G(\lambda)}{n} \geq \frac{\lambda}{1+\lambda} \cdot \frac{W(d \log(1+\lambda))}{d \log(1+\lambda)},$$

where $W(z) e^{W(z)} = z$.

Pf: • $\mathbb{E}|I| \overset{①}{=} \sum_{v \in V(G)} \Pr(v \in I)$



$$\overset{②}{\geq} \frac{1}{d} \sum_{v \in V(G)} \sum_{u \in N(v)} \Pr(u \in I)$$

• Def: v is suitable if $N(v) \cap I = \emptyset$

$$\Pr(v \in I) = \Pr(\{v \in I\} \wedge \{v \text{ suitable}\})$$
$$= \Pr(v \in I \mid v \text{ suitable}) \cdot \Pr(v \text{ suitable})$$

• $\Pr(v \in I \mid v \text{ suitable}) = \frac{\lambda}{1+\lambda}$ (Spatial Markov prop)

$\Downarrow$

$\Pr(v \in I \mid V(G) - v)$

$= \Pr(v \in I \mid N(v))$

• Def X_v r.v. = # suitable neighbrs of v

$X_v \in [0, d]$

indep $\Leftarrow$ Δ-free

$$\Rightarrow \Pr(v \text{ suitable}) = \sum_{x=0}^{d} \Pr(X_v = x) \left(\frac{1}{1+\lambda}\right)^x$$
$$= \mathbb{E}\left(\frac{1}{1+\lambda}\right)^{X_v}$$

• Now, $\mathbb{E}|I| = \sum_{v \in V(G)} \frac{\lambda}{1+\lambda} \cdot \mathbb{E}\left(\frac{1}{1+\lambda}\right)^{X_v}$

occup. frac

$$\frac{1}{n}\mathbb{E}|I| = \frac{\lambda}{1+\lambda} \frac{1}{n} \sum_{v \in V(G)} \mathbb{E}\left(\frac{1}{1+\lambda}\right)^{X_v}$$

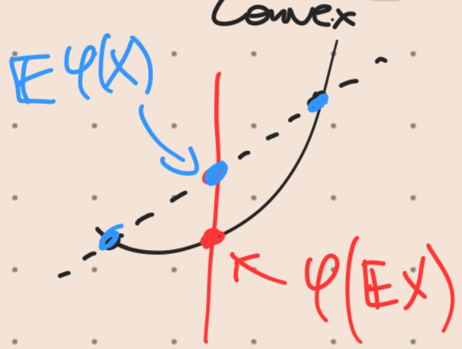
- Two layer of randomness $\begin{cases} I \sim \text{Hard-core dist.} \\ v \sim_{\text{unif}} V(G) \end{cases}$

$$\Rightarrow \frac{1}{n}\mathbb{E}|I| = \frac{\lambda}{1+\lambda}\,\mathbb{E}_{I,v}\left(\frac{1}{1+\lambda}\right)^{X}, \text{ where}$$

$X \overset{\text{r.v.}}{=}$ # suitable neighbors of a unif random v

[Recall Jensen's ineq.

$$\mathbb{E}\varphi(X) \geq \varphi(\mathbb{E}X)$$



$$\Rightarrow \frac{1}{n}\mathbb{E}|I| \geq \frac{\lambda}{1+\lambda}\left(\frac{1}{1+\lambda}\right)^{\mathbb{E}X}$$

- ② $$\mathbb{E}|I| \geq \frac{1}{d}\sum_{v\in V(G)}\sum_{u\in N(v)} \underbrace{\Pr(u\in I \mid u \text{ suitable})}_{=\frac{\lambda}{1+\lambda}} \cdot \Pr(u \text{ suitab}$$

$$\Rightarrow \frac{1}{n}\mathbb{E}|I| \geq \frac{1}{d}\cdot\frac{\lambda}{1+\lambda}\cdot \boxed{\frac{1}{n}\sum_{v\in V(G)}\sum_{u\in N(v)}\Pr(u \text{ suitable})}$$

$$= \frac{1}{d}\frac{\lambda}{1+\lambda}\,\mathbb{E}X$$

- So we have from the two bounds that

$$\frac{1}{n}\bar{\alpha}_G(\lambda) \geq \frac{\lambda}{1+\lambda}\cdot\max\left(\underbrace{\left(\frac{1}{1+\lambda}\right)^{\mathbb{E}X}}_{① \searrow},\ \underbrace{\frac{\mathbb{E}X}{d}}_{② \nearrow}\right)$$

Optimized when $\left(\frac{1}{1+\lambda}\right)^x = \frac{x}{d}$ — plug in

$$\cdots\cdots \Rightarrow \quad x = \frac{W(d\log(1+\lambda))}{\log(1+\lambda)}$$

$\blacksquare$

Conseq.

(i) $\bar{\alpha}_G(\lambda)$ monotone increasing in λ

$$\bar{\alpha}(G) = \bar{\alpha}_G(1) \geq \bar{\alpha}_G\left(\frac{1}{\log d}\right) = \cdots = (1+o(1))\frac{\log d}{d}\cdot n$$

(ii) counting result.

Recall: $\bar{\alpha}_G(\lambda) = \lambda\left[\log P_G(\lambda)\right]'$

$$\Rightarrow \quad \frac{1}{n}\log P_G(1) = \int_0^1 \frac{\bar{\alpha}_G(t)}{t}\,dt$$

$$\overset{\text{Thm}}{\geq} \frac{1}{d}\int_0^1 \frac{1}{1+t}\,\frac{W(d\log(1+t))}{\log(1+t)}\,dt$$

$$\overset{u = W(d\log(1+t))}{=} \frac{1}{d}\int_0^{W(d\log 2)} (1+u)\,du = \cdots = \left(\frac{1}{2}+o_d(1)\right)\frac{\log^2 d}{d}$$

Cor: $\forall$ n-vx Δ free w/ max deg $\leq d$

$$i(G) \geq e^{\left(\frac{1}{2}+o_d(1)\right)\frac{\log^2 d}{d}\cdot n}.$$

\# indep sets

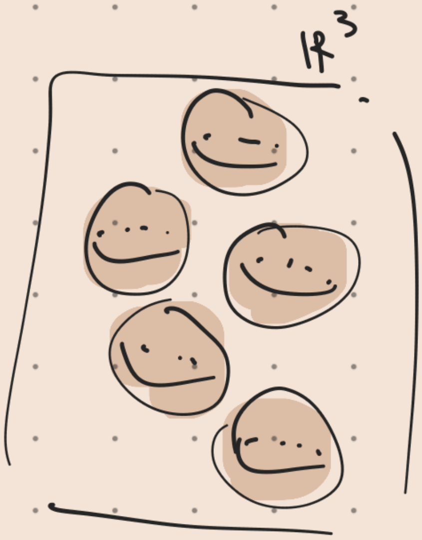
- Stat. phy = study of matter via prob. & stat. methods

<u>Motivation</u>: Whether (global) macroscopic properties of matter can be derived solely from their local microscopic interactions.

<u>Instead of</u>	
keeping track of all particles	treat them as randomly distributed w/ certain local constraints.

	<u>Global</u>	<u>Local</u> constraint
Indep set	$\alpha(G)$	edges $u \sim v$
sphere packing	packing density	centers of balls not so close (distance $\geq 2r_d$)

§ Sphere packing in $\mathbb{R}^d$



<u>Def</u>: • $B_R(0)$ = radius-R ball in $\mathbb{R}^d$ centered at the origin.

• r_d = radius s.t. $B_{r_d}(0)$ has volume 1.

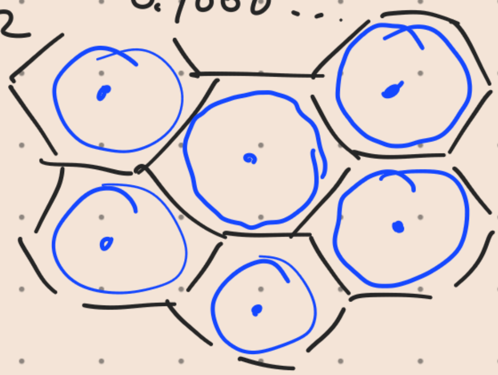
- Write $\mathcal{P}$ for a sphere packing of non-overlapping identical spheres in $\mathbb{R}^d$.

- sphere packing density in $\mathbb{R}^d$

$$\theta(d) := \sup_{\mathcal{P}} \limsup_{R\to\infty} \frac{\mathrm{vol}(\mathcal{P} \cap B_R(0))}{\mathrm{vol}(B_R(0))}$$

<u>Known</u>:
- $\theta(1) = 1$
- 1892 Thue $\theta(2) = \pi/\sqrt{12} = 0.9068\ldots$
- 2005 Hales $\theta(3) = \frac{\pi}{\sqrt{18}}$ $= 0.7404\ldots$

(orange packing is optimal)



- 8, 24-dim known.

We want to understand $\theta(d)$, $d\to\infty$.

- A set S, $k \in \mathbb{N}$, $\binom{S}{k}$ = set of all k-sets in S

S^k = all ordered k-tuples in S

$\{s_1, \ldots, s_k\}$

• Let $S \subseteq \mathbb{R}^d$ measurable set.

<u>Def</u> $P_k(S) \subseteq \binom{S}{k}$

$$P_k(S) := \{ (x_1, \dots, x_k) : d(x_i, x_j) > 2r_d \ \ \forall i,j \}$$

i.e. $P_k(S)$ = the set of all size-k sphere packing in S.

• Canonical Hard sphere model

Partition function $\hat{Z}_S(k) = \frac{1}{k!} \int_{S^k} \mathbb{1}_{D(x_1, \dots, x_k)} \, dx_1 \cdots dx_k$.

where $D(x_1, \dots, x_k)$ = event that $d(x_i, x_j) > 2r_d$

• Note that, $\hat{Z}_S(k)$ = volume of $P_k(S)$.

a unif random k-tuple X_k • $\mathbb{P}(X_k \in P_k(S)) = \frac{\hat{Z}_S(k)}{\text{total vol.}}$

$\left(\text{total vol} = \frac{\text{vol}(S)^k}{k!} \right)$ $= \frac{k!}{\text{vol}(S)^k} \cdot \hat{Z}_S(k)$.

• <u>Def</u> Grand canonical Hard sphere model at S, w/ fugacity λ

Partition funct.

$$\hat{Z}_S(\lambda) = \sum_{k=0}^{\infty} \lambda^k \, \underline{\hat{Z}_S(k)}$$

(continuous version of Hard-core distribution)

$$Z_G(\lambda) = P_G(\lambda) = \sum_{k=0}^{\infty} \underline{i_k(G)} \cdot \lambda^k,$$

$i_k(G)$ = # indep. sets in G of size k.