Lecture 28

Previously ...

- Perron-Frobenius thm. (symm)

  G connected, A adj matr. $\alpha_1 \geq \cdots \geq \alpha_n$

  $\Rightarrow$ (i) α_1 has a positive eigenvector
  (ii) $\forall i$, $\alpha_i \in [-\alpha_1, \alpha_1]$
  (iii) $\alpha_1 > \alpha_2$

  Cor. $\forall G$ connected, if a positive vector x is an eig-vect. of A, then the corresp. eig-value is α_1.

- Random walk
  - transition matrix $P = D^{-1}A \overset{d\text{-reg}}{=} \frac{1}{d} A$
  - $P(i,j) = \mathbb{P}(i \to j)$
  - $P^t(i,j) = \mathbb{P}$(starting at i and arrive at j after t steps)

$$\boxed{(P^T)^t \cdot p_0 = p_t}$$

- Stationary distribution: $P^T \cdot \pi = \pi$

$$\pi = \frac{\mathbf{d}}{\sum_{i=1}^{n} d(v_i)} \qquad \pi(v_i) \propto d(v_i)$$

$$\overset{\text{reg}}{=} \frac{1}{n} \cdot \mathbb{1} \quad \text{uniform for regular graph.}$$

Q: Does random walk $\rightsquigarrow$ stationary distribution

if yes, how fast?

We shall see a connection btw this & spectral gap of the transition matrix P.

Spectrum of P

- $P = D^{-1}A$ in general is not symm.

  So we don't orthogonal eigenvectors for P.

Def X is similar w/ Y if $X = Q^{-1} Y Q$. Similar matrices have the same set of eigenvalues.

- P is similar to the symm. matrix $\tilde{A} = D^{-1/2} A D^{-1/2}$

$$P = D^{-1/2} \cdot \tilde{A} \cdot D^{1/2} \quad \text{or} \quad \tilde{A} = D^{1/2} P D^{-1/2}.$$

Lem (G connected)

	eig vect.	eig value
$\tilde{A}$:	x	α
$\Leftrightarrow$ P:	$y = D^{-1/2}x$	α

Pf: $P \cdot y = D^{-1/2} \tilde{A} \cancel{D^{1/2}} \cdot \cancel{D^{-1/2}} x = D^{-1/2} \cdot \alpha \cdot x = \alpha \cdot y$ ∎

Lem: The deg. vect. d is a Perron vector of P^T w/ eigenvalue 1.

Pf: $d^T \cdot P = d^T \cdot D^{-1} A = 1^T \cdot A = d^T$ ▨

Cor: Spectrum of P is in $[-1, 1]$

Def: Lazy random walk $\begin{cases} 1/2 & \text{stay put} \\ 1/2 & \text{do the same as random walk.} \end{cases}$

$$\tilde{P} = P/2 + I/2$$

spectrum $\in [0, 1]$

Rmk:
- Lazy random walk converges on all graphs.
- all eigenvalue nonnegative, convergence (of the random walk) only depends on α_2 or the spectral gap $1 - \alpha_2$

We shall prove that $\forall$ non-bipartite connected graph G, the random walk (w./ any initial distr. p_0) converges to the stationary distr. π.

Rmk:
- Not true for bipartite graphs b/c of the parity issue $p_t \not\to \pi$
- Can consider instead lazy random walk for bip. graphs.

We shall see that in fact the rate of conv. is exponential when having spectral gap.

Def Mixing rate of a random walk is

$$\mu = \limsup_{t\to\infty} \max_{i,j} \left| \underbrace{P^t(i,j)}_{\mathbb{P}(\text{start at } i \rightsquigarrow j \text{ at time } t)} - \pi(j) \right|^{1/t}$$

$\pi(v_j)$

Thm: Let G be a graph w/ transition matrix P and eigenvalues $\alpha_1 \geq \dots \geq \alpha_n$.

$\Longrightarrow$ Then $\forall$ starting vx i and any vertex j $\forall t > 0$

$$\left| P^t(i,j) - \pi(j) \right| \leq \sqrt{\frac{\pi(j)}{\pi(i)}} \cdot \mu^t ,$$

where $\mu = \max\{|\alpha_2|, |\alpha_n|\}$.

HW More generally, $\forall S \subseteq V(G)$,

$$\left| \mathbb{P}\left(\begin{matrix}\text{starting at } i \\ \text{end in } S \text{ at time } t\end{matrix}\right) - \pi(S) \right| \leq \sqrt{\frac{\pi(S)}{\pi(i)}} \cdot \mu^t .$$

If G is connected $\Rightarrow \alpha_2 < \alpha_1$, if non-bip. $\Rightarrow \alpha_n > -1$

Cor: For connected, non-bipartite G mixing rate is $\mu \leq \max\{|\alpha_2|, |\alpha_n|\}$.

Pf • Recall $P = D^{-1}A = D^{-1/2}\tilde{A}D^{1/2}$, $\tilde{A} = D^{-1/2}AD^{-1/2}$

• As $\tilde{A}$ is symm., we can write it as

$$\tilde{A} = \sum_{k=1}^{n} \alpha_k v_k \cdot v_k^T \quad \text{, where } v_1, \dots, v_k \text{ orthonormal}$$

w/ eig value $\alpha_1 \geq \dots \geq \alpha_n$.

$$\Rightarrow \tilde{A}^t = \sum_{k=1}^{n} \alpha_k^t v_k v_k^T$$

• $$P^t = \left(D^{-1/2}\tilde{A}D^{1/2}\right)^t = \left(D^{-1/2}\tilde{A}\cancel{D^{1/2}}\right)\cdot\left(\cancel{D^{-1/2}}\tilde{A}D^{1/2}\right)\cdots(\dots)$$

$$= D^{-1/2}\tilde{A}^t D^{1/2}$$

$$= \sum_{k=1}^{n} \alpha_k^t \cdot D^{-1/2} v_k \cdot v_k^T D^{1/2}$$

$$\overset{i}{(0,0,\dots,1,0,0)} \begin{pmatrix} \frac{1}{\sqrt{d(v_1)}} & & \\ & 0 & \\ & & \frac{1}{\sqrt{d(v_n)}} \end{pmatrix} = \frac{1}{\sqrt{d(v_i)}} e_i^T$$

So $$P^t(i,j) = e_i^T P^t e_j = \sum_{k=1}^{n} \alpha_k^t \underbrace{e_i^T D^{-1/2}} v_k v_k^T \underbrace{D^{1/2} e_j}_{\sqrt{d(v_j)}\cdot e_j}$$

$$\sqrt{\frac{\pi(j)}{\pi(i)}} \qquad = \sqrt{\frac{d(v_j)}{d(v_i)}} \sum_{k=1}^{n} \alpha_k^t e_i^T v_k \cdot v_k^T e_j$$

$\boxed{\alpha_1 = 1}$

1st term in sum: $\cancel{\alpha_1^t} \cdot e_i^T v_1 \cdot v_1^T e_j$

$$= \sqrt{\pi(i)} \cdot \sqrt{\pi(j)}$$

HW

$v_1 = \pi^{1/2}$

$v_1(i) = \sqrt{\pi(i)}$

i.e. $\pi^{1/2}$ is a perron eigenvector of P.

$\Rightarrow$ 1st term $= \pi(j)$

$$\left|P^t(i,j) - \pi(j)\right| \leq \sqrt{\frac{\pi(j)}{\pi(i)}} \left| \underbrace{\sum_{k=2}^{n} \alpha_k^t e_i^T v_k \cdot v_k^T e_j}_{\leq \mu^t} \right|$$

$$, \mu = \max\{|\alpha_2|, |\alpha_n|\}$$

$$\left|\sum \cdots\cdots\right| \le \mu^t \sum_{k=\cancel{2}1}^{n} \left|e_i^T v_k\right| \cdot \left|v_k^T e_j\right|$$

C-S

$$\le \mu^t \sqrt{\sum_{k=1}^{n} \left|e_i^T v_k\right|^2} \cdot \sqrt{\sum_{k=1}^{n} \left|v_k^T e_j\right|^2}$$

as $v_1 \cdots v_n$ are orthonormal

$$= \mu^t \|e_i\| \cdot \|e_j\| = \mu^t . \qquad \blacksquare$$

<u>Rmk</u>. Usually by considering the lazy random walk (so $\tilde{P} = P/2 + I/2$ & spectrum $\in [0,1]$), mixing rate is only related to α_2 or the spectral gap $1-\alpha_2$.

<u>Next</u>: $\left\{ \begin{array}{l} \text{Total variation distance} \\ \text{mixing time.} \end{array} \right.$