Lecture 27

Previously

- Cheeger's ineq $\quad \nu_2/2 \le \phi(G) \le \sqrt{2\nu_2} \overset{d\text{-reg}}{=} \sqrt{2 \min_{x \perp 1} R_N(x)}$

  Robust version: $\forall x \perp 1$, Fiedler's Alg $\xrightarrow{\text{cut}} (S, S^c)$

$$\phi(S, S^c) \le \sqrt{2R_N(x)}$$

- Harper: $\quad \phi(Q_d) \ge \frac{1}{d} \qquad Q_{d-1} \subseteq Q_d$

- normalised Laplacian $\quad N = D^{-1/2} L D^{-1/2} = I - \underbrace{D^{-1/2} A D^{-1/2}}_{\text{normalised adj mat.}}$

Recall $\quad M$ real symm $\xrightarrow{\text{spectral thm}}$ ~~orthogonal~~ normal $v_1, \dots, v_n$

$\lambda_1, \dots, \lambda_n$

Cor (eig decomposition) $\quad M$ real symm w/ $\uparrow \qquad \Lambda = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$

$$\Rightarrow \quad M = V \Lambda V^T = \sum_{k=1}^{n} \lambda_k v_k v_k^T, \quad \text{where } V = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix}$$

Pf:

$$MV = M \begin{pmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{pmatrix} = \begin{pmatrix} | & & | \\ \lambda_1 v_1 & \cdots & \lambda_n v_n \\ | & & | \end{pmatrix} = V \cdot \Lambda$$

$$\Rightarrow \quad M = V \Lambda V^{-1}$$

Note that $V^{-1} = V^T$. $\blacksquare$

Thm (Perron-Frobenius symm version)

Let G be a connected graph w/ adj mat. A & eig values $\alpha_1 \ge \dots \ge \alpha_n$.

Then (i) The largest eig value α_1 has a positive eig. vector.

(ii) $\forall i$, $\alpha_i \in [-\alpha_1, \alpha_1]$

(iii) $\alpha_1 > \alpha_2$

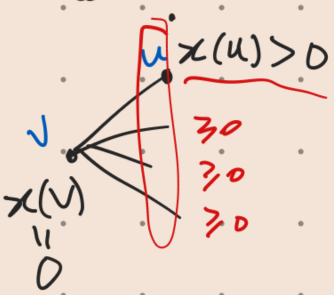
<u>Lem 1</u>. G connected, A adj mat.

if x is a non-negative eig vect,

$\Rightarrow$ x has to be a positive vector.

<u>Pf</u>: Supp not $\Rightarrow \exists\, uv \in E$ s.t.

$\Rightarrow$



$$0 < \sum_{u \in N(v)} x(u) = (Ax)_v = (\alpha x)_v = 0 \quad ↯ \quad \square$$

<u>Pf</u> (P-F symm) Let $x_1, \ldots, x_n$ orthonormal
$\alpha_1, \ldots, \alpha_n$

Let $y \in \mathbb{R}^V_{\geq 0}$ be s.t.

$$y(v) = |x_1(v)| \Rightarrow y^T y = x_1^T x_1 = 1$$

We claim that y is an eig. vector of α_1

then by Lem 1 $\Rightarrow$ $y \in \mathbb{R}^V_{>0}$ is a positive eig vect.

$$\alpha_1 = x_1^T A x_1 = \sum_{u,v} x_1(u) A(u,v) x_1(v)$$

$$\le \sum |x_1(u)| A(u,v) |x_1(v)| = y^T A y = R_A(y) \overset{C\text{-}F}{\le} \alpha_1$$

$\Rightarrow$ y is an eig vect. of α_1.

For (ii), suffices to show that $\alpha_n \geq -\alpha_1 \Leftrightarrow |\alpha_n| \le \alpha_1$

Let $z \in \mathbb{R}^V_{\geq 0}$: $z(v) = |x_n(v)|$

$$|\alpha_n| = |x_n^T A x_n| \le \left| \sum_{u,v} x_n(u) A(u,v) x_n(v) \right|$$

$$\overset{\Delta\text{-ineq}}{\le} \sum_{u,v} |x_n(u)| A(u,v) |x_n(v)| = z^T A z \overset{C\text{-}F}{\le} \alpha_1.$$

For (iii), $\alpha_1 > \alpha_2$. Let $w \in \mathbb{R}^V_{\geq 0}$: $w(u) = |x_2(u)|$

$$\alpha_2 = x_2^T A x_2 \le w^T A w \overset{C\text{-}F}{\le} \alpha_1$$

$\Downarrow$

w is a non-neg. eig vect. of α_1

If $\alpha_2 = \alpha_1$.

Lem 1 $\Rightarrow$ w is positive vector.

$\Rightarrow$ x_2 has no zero entry

$$\boxed{\text{• As } G \text{ is connected and } x_2 \perp x_1 \text{ (assume } x_1 \text{ positive by Part (i))}}$$ $\Rightarrow \exists\, uv \in E$

$x_2(u) > 0$ $\quad$ $x_2(v) < 0$

Such u,v contributes negative to $x_2^T A x_2$ $\Rightarrow$ $x_2^T A x_2 < w^T A w$ $\Rightarrow \alpha_2 < \alpha_1$ ∎

<u>Cor</u> $\forall$ connected G, if a positive vector x is an eigenvector of A (adj mat)

$\Rightarrow$ the corresponding eigenvalue is α_1.

<u>Pf</u>: • Say $Ax = \alpha_i x$ for some $i \in [n]$

• P-F thm $\Rightarrow$ $\exists\, y \in \mathbb{R}^V_{>0}$ eig vect. of α_1

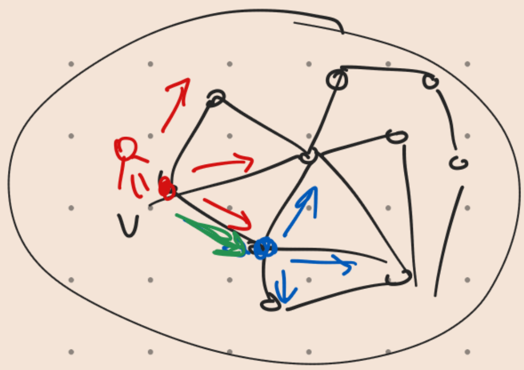
• A is symm $\Rightarrow$

$$\langle Ax, y\rangle = \langle x, Ay\rangle$$
$$\alpha_i x^T y = \alpha_1 x^T y$$

as $x^T y > 0$ $\Rightarrow$ $\alpha_i = \alpha_1$ ∎

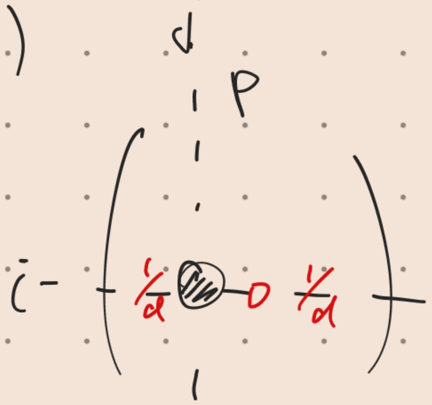
Random Walk

• Random walk on a graph G is a random process that starts at a vertex in G, at each step, choose a unif random neighbour to move to.



• appear
- card shuffling
- Brownian motion of a particle.

Transition matrix $(P)_{ij} = P(i,j)$



$P(i,j) = \mathbb{P}(i \to j)$

- d-reg $\quad P = \frac{1}{d} A$
- general $\quad P = D^{-1}A = \begin{array}{c} \\ i \cdots \end{array} \begin{pmatrix} \frac{1}{d(v_1)} & \cdots & \frac{1}{d(v_1)} \\ & & \\ \frac{1}{d(v_i)} & \cdot & \frac{1}{d(v_i)} \end{pmatrix} \quad D = \begin{pmatrix} d(v_1) & & 0 \\ & \ddots & \\ 0 & & d(v_n) \end{pmatrix}$

$j \in N(i)$

$$P(i,j) = \begin{cases} \frac{1}{d(v_i)}, & j \in N(i) \\ 0, & \text{o.w} \end{cases}$$

$\mathbb{P}(\text{start at } v)$

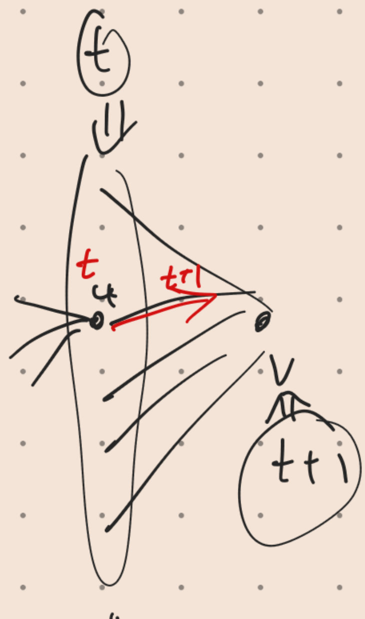
Consider initial distribution $p_0 = \begin{pmatrix} p_0(v) \\ \vdots \\ \vdots \end{pmatrix}$

and write p_t for the distribution at time t.

$$\boxed{p_t^T = p_0^T \cdot P^t \Leftrightarrow (P^T)^t \cdot p_0 = p_t}$$

Suffices to show

$$p_{t+1}^T = p_t^T \cdot P$$



$$(p_{t+1})_v = p_{t+1}(v) = \mathbb{P}\left(\begin{matrix}\text{arrive at } v \\ \text{at time } t+1\end{matrix}\right)$$

$$= \sum_{u \in N(v)} \mathbb{P}\left(\begin{matrix}\text{at } u \\ \text{time } t\end{matrix}\right) \cdot \mathbb{P}(u \to v)$$

$$= \sum_{u \in N(v)} p_t(u) \cdot P(u,v) = (P^T \cdot p_t)_v$$

$N(v)$

Recall $P(i,j) = \mathbb{P}(\underset{\text{in one step}}{i \to j})$

$$P^t(i,j) = \mathbb{P}\begin{pmatrix}\text{start at } i \\ \text{and arrive} \\ \text{at } j \text{ at time } t\end{pmatrix}$$

$$p_0^T \cdot P^t = \delta_i^T \cdot P^t = p_t^T$$

$$p_0 = \delta_i = \begin{pmatrix}0\\ \vdots\\ 1\\ \vdots \\ 0\end{pmatrix} \cdots i$$

$P^t(i,j)$

<u>Def</u> Stationary distribution π for a random walk w/ transition matrix P is

$$P^T \cdot \pi = \pi$$

$$\mathbf{d} = \begin{pmatrix} d(v_1) \\ \vdots \\ d(v_n)\end{pmatrix}$$

Exer $\pi = \dfrac{\mathbf{d}}{\sum_{i=1}^n d(v_i)} = \dfrac{\mathbf{d}}{\mathbb{1}^T \cdot \mathbf{d}}$

- At stationary dist., every vertex is visited w/ Prob. proportional to its deg.
- if G is regular $\Rightarrow \pi = \frac{1}{n}\mathbb{1}$

$\boxed{\text{NEXT TIME}}$ $\forall$ non bip $\wedge$ connected $G \rightsquigarrow \pi$ (w./ any p_0)