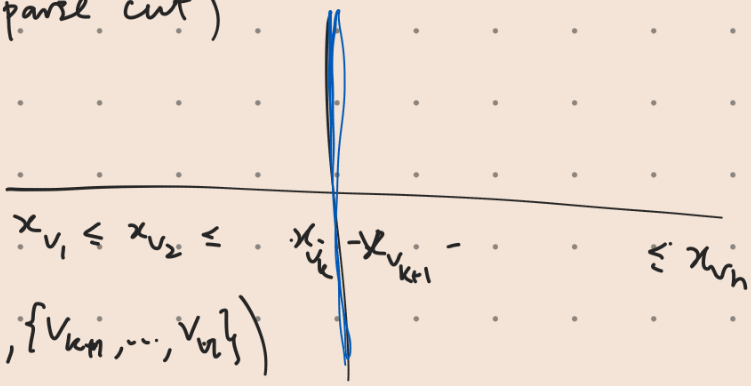
Lecture 26

Recall:

- Conductance: n-vx d-reg G, $N = I - \frac{1}{d}A$
  - $\phi(S) = \frac{|\partial S|}{d|S|} = R_N(\mathbb{1}_S)$, $\phi(S, S^c) = \max\{\phi(S), \phi(S^c)\}$
  - $\phi(G) = \min_{|S| \le n/2} \phi(S) = \min_S \phi(S, S^c)$
- $\phi(G) \ge \nu_2 / 2$ → algebraic connectivity
- Fiedler's alg. (for sparse cut)
  - sort
  - cut

$\min_k \phi(\{v_1, \dots, v_k\}, \{v_{k+1}, \dots, v_n\})$



Thm [Cheeger's ineq]

$$\nu_2/2 \le \phi(G) \le \sqrt{2\nu_2} = \sqrt{2 \min_{x \perp \mathbb{1}} R_N(x)}$$

- By C-F: $\nu_2 = \min_{\substack{x \ne 0 \\ x \perp \mathbb{1}}} R_N(x) = \min_{\substack{x \ne 0 \\ x \perp \mathbb{1}}} \frac{\sum_{uv \in E} (x_u - x_v)^2}{\|x\|^2}$

  d-reg

  $\sum_{v \in V} x_v^2$

Thm (Robust version) Let $x \perp \mathbb{1}$ and (S, S^c) be the minimiser in Fiedler's alg. w/ input x.

$$\Rightarrow \phi(S, S^c) \le \sqrt{2 R_N(x)}$$

$= \max\{\phi(S), \phi(S^c)\}$

Lem1 $\forall\, y \in \mathbb{R}^V_{\geq 0}$, $\exists\ t > 0$ s.t.

$$\phi(S_t) = \phi(\{v : y_v \geq t\}) \leq \sqrt{2R_N(y)}.$$

- $\phi(S, S^c) = \phi(S) \iff |S| \leq n/2$

Lem2 Let $x \perp \mathbb{1}$. Then $\exists\ y \in \mathbb{R}^V_{\geq 0}$ s.t.

(i) # positive entries in $y \leq n/2$

(ii) $R_N(y) \leq R_N(x)$

(iii) Cuts considered in Fiedler's alg. w/ input y are the same as those w/ input x.

Pf [Robust Cheeger]. Take y from Lem 2, and let S_t be the set returned from Lem 1

$$\phi(S_t) = \phi(\{v : y_v \geq t\}) \overset{\text{Lem1}}{\leq} \sqrt{2R_N(y)}$$

$$\overset{\text{Lem2 (ii)}}{\leq} \sqrt{2R_N(x)}.$$

On the other hand,

$$\phi(S_t) \overset{\text{Lem2(i)}}{=} \phi(S_t, S_t^c) \overset{\text{Lem2(iii)}}{\geq} \phi(S, S^c)$$

Pf (Lem2) Note that $\forall\, c \in \mathbb{R}$, we have

$$R_N(x + c\mathbb{1}) \leq R_N(x) = \frac{x^T N x}{\|x\|^2}$$

$I - \frac{1}{2}A$

$N \cdot \mathbb{1} = 0$, so numerator same

$$\|x + c\mathbb{1}\|^2 = \|x\|^2 + \|c\mathbb{1}\|^2 \qquad x \perp \mathbb{1}.$$

- Let m = median value of entries of x.

def: $z = x - m \cdot \mathbb{1}$

so $R_N(z) \le R_N(x)$

Note that z has $\le n/2$ ⊕ entries

...... ⊖ ......

$\le n/2$ | $\le n/2$

$- - - - - - 0 \; + + + + + + + + +$

u ⊖ | v ⊕

Write $z = z^+ - z^-$,
$$z^+_v = \begin{cases} x_v & v \oplus \\ 0 & \text{o.w.} \end{cases}$$
$$z^-_u = \begin{cases} -x_u & u \ominus \\ 0 & \text{o.w.} \end{cases}$$

- Note that cuts in F's alg. w/ input z^+ or z^- same as those w/ input x.

- Left to show

$$R_N(z) \ge \min\{R_N(z^+), R_N(z^-)\}$$

$$R_N(z) = \frac{\sum_{uv \in E}(z_u - z_v)^2}{\|z\|^2} = \frac{\sum_{uv \in E}\left((z^+_u - z^+_v) - (z^-_u - z^-_v)\right)^2}{\|z^+\|^2 + \|z^-\|^2}$$

Claim: $\left((z^+_u - z^+_v) - (z^-_u - z^-_v)\right)^2 \ge (z^+_u - z^+_v)^2 + (z^-_u - z^-_v)^2$

$$\ge \frac{\sum_{uv}\left[(z^+_u - z^+_v)^2 + (z^-_u - z^-_v)^2\right]}{\|z^+\|^2 + \|z^-\|^2}$$

$$= \frac{R_N(z^+)\cdot\|z^+\|^2 + R_N(z^-)\|z^-\|^2}{\|z^+\|^2 + \|z^-\|^2}$$

Pf (Claim).

- $x_u, x_v \ge 0$ or $x_u, x_v \le 0$ } "=" holds

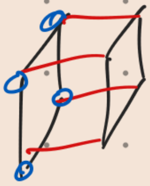
- $x_u > 0$, $x_v < 0$ } $\Rightarrow$ LHS $= (z^+_u + z^-_v)^2$, RHS $= (z^+_u)^2 + (z^-_v)^2$ ∎

$\geq \min \{ R_N(Z^+), R_N(Z^-) \}$.

— Application for isop. in hypercube —

Q_d : d-dim hypercube

	eig. $0 \leq i \leq d$	multiplicity
λ_i	$2i$	$\binom{d}{i}$
ν_i	$2i/d$	$\binom{d}{i}$



<u>Thm</u> (Harper 76) $\theta(Q_d) \geq 1$.

S

<u>Pf</u>: $\lambda_2 = 2$ and $\theta(G) \geq \lambda_2/2$.

$\theta(S) = \frac{|\partial S|}{|S|}$

Tight: consider a copy of Q_{d-1} inside.

<u>Conductance</u> $\Phi(Q_d) \geq \nu_2/2 = 1/d$

Using multicommodity flow method,

<u>Thm</u> [Babai - Szegedy 92]

G connected, edge-transitive w/ diameter D

$\Rightarrow \Phi(G) \geq 1/D$

$\operatorname{diam}(Q_d) = d$

—— Normalised Laplacian for general graphs —

- $L = D - A$

$D = \begin{pmatrix} d(v_1) & & 0 \\ & \ddots & \\ 0 & & d(v_n) \end{pmatrix}$

$$N = D^{-1/2} L D^{-1/2}$$
$$= D^{-1/2}(D-A)D^{-1/2}$$
$$= I - D^{-1/2} A D^{-1/2}$$

$$D^{-1/2} = \begin{pmatrix} \frac{1}{\sqrt{d(v_1)}} & & 0 \\ & \ddots & \\ 0 & & \frac{1}{\sqrt{d(v_n)}} \end{pmatrix}$$

$\boxed{\text{Why?}}$

Recall that for d-reg graph, $N = I - \frac{1}{d} A$

and
- $\nu_i \in [0, 2]$
- multiplicity of 0 = # conn. comp.
- $\nu_n = 2 \iff \exists$ bip. component.

For d-regular graphs

$$R_N(x) = \frac{x^T N x}{x^T x} = 2 - \frac{\sum_{uv \in E} (x_u + x_v)^2}{\underline{d} \cdot \sum_v x_v^2}$$

Suggests $\Rightarrow$ $2 - \dfrac{\sum_{uv \in E} (x_u + x_v)^2}{\sum_{v \in V} d(v) \cdot x_v^2}$

$$\ldots = \frac{\sum_{uv \in E} (x_u - x_v)^2}{\sum_{v \in V} d(v) \cdot x_v^2}$$

$$= \frac{x^T L x}{x^T D x} = \frac{\cdots\cdots}{x^T D^{1/2} \cdot D^{1/2} x}$$

$$R_M(z) = \frac{z^T M z}{z^T z}$$

$$\| D^{1/2} x \|^2 = x^T D x$$

Need $x^T L x = (x^T D^{1/2}) \cdot D^{-1/2} L D^{-1/2} (D^{1/2} x)$

$$= R_{D^{-1/2} L D^{-1/2}} \left(D^{1/2} x \right) = \frac{x^T L x}{x^T x}$$

The point is that the map

$$x \longmapsto D^{1/2} x \quad \text{is bijective}$$

- $\nu_1 \le \cdots \le \nu_n$ for $N = D^{-1/2} L D^{-1/2}$

$$\nu_k = \min_{\dim U = k} \max_{\substack{x \in U \\ x \neq 0}} R_N(x)$$

$$= \min_{\dim U = k} \max_{\substack{x \in U \\ x \neq 0}} R_N(D^{1/2} x)$$

$$= \cdots\cdots \frac{x^T L x}{x^T D x}.$$

d

- $d^{1/2} = \begin{pmatrix} \sqrt{d(v_1)} \\ \vdots \\ \sqrt{d(v_n)} \end{pmatrix}$ is an eig. vector for 0.

$$N d^{1/2} = D^{-1/2} L \underbrace{D^{-1/2} \cdot d^{1/2}}_{\mathbb{1}} = D^{-1/2} L \cdot \mathbb{1} = 0$$

CF

- $$\nu_2 = \min_{\substack{x \perp d^{1/2} \\ x \neq 0}} \frac{x^T N x}{x^T x} \overset{\text{HW}}{=} \min_{\substack{y \neq 0 \\ y \perp d}} \frac{y^T L y}{y^T D y}$$

Recall $\quad \phi(S) = \dfrac{|\partial S|}{d(S)} \longrightarrow$ deg sum in S

Def : volume of S = $vol(S) = \sum_{v \in S} d(v)$

$$\phi(S) = \frac{|\partial S|}{vol(S)}$$

$$\phi(S, S^c) = \max\{\phi(S), \phi(S^c)\}$$

$$\phi(G) = \min_{|S| \le n/2} \phi(S) = \min_{S} \phi(S, S^c).$$

Thm (Cheeger ineq) true also for irreg graphs.

$$\nu_2/2 \le \phi(G) \le \sqrt{2\nu_2}$$