Lecture 24

Recall : d-reg , normalised Laplacian

$$N = \frac{1}{d} L = I - \frac{1}{d} A$$

- $R_N(x) = \frac{x^T N x}{x^T x} = \frac{\sum_{uv \in E} (x_u - x_v)^2}{d \sum_{v \in V} x_v^2}$
- $0 = \nu_1 \leq \nu_2 \leq \cdots \leq \nu_n \leq 2$
  - $\nu_n = 2 \iff \exists$ bipartite comp.
  - multiplicity of 0 = # conn. comp. in G.

- We have seen spectral gap of adjacency matrix which is equiv. to discrepancy property for quasirandom graphs & Cayley graphs of finite abelian groups.

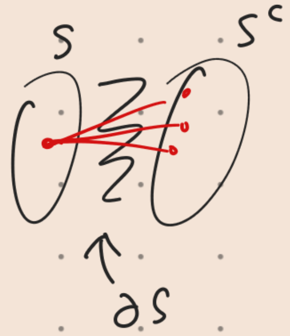
- For (normalised) Laplacian, as the 1^{st} eigenvalue $\lambda_1 = 0$, the spectral gap is precisely the value of λ_2.

We shall see the larger λ_2 (spe. gap) the better expansion property.

Isoperimetry & 2^{nd} eigenvalue

<u>Def</u> • The boundary of a set $S \subseteq V$ is

$$\partial S = E_G(S, V \setminus S) \quad , \quad V\setminus S = S^c$$
$$= E_G(S, S^c)$$



• The isoperimetric number of $S \subseteq V$ is

$$\theta(S) = \frac{|\partial S|}{|S|}$$

basically the average degree out of S

$$\sum_{uv \in E} \left((1_S)_u - (1_S)_v\right)^2 = 1_S^T L 1_S$$

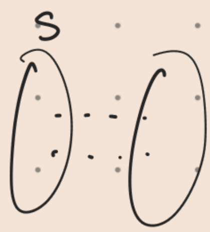
Note that $\theta(S) = \dfrac{e(S, S^c)}{|S|} = R_L(1_S)$

$1_S^T 1_S$

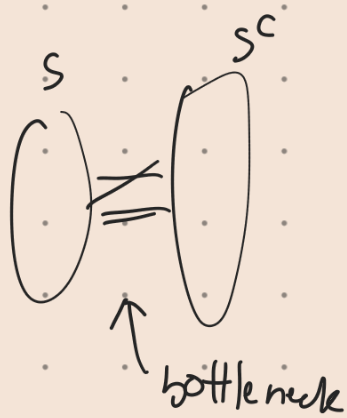
<u>Def</u>: The isoperimetric number, or Cheeger constant of a graph G is

$$h(G) = \min_{0 < |S| \leq n/2} \theta(S)$$
$$= \min \left\{ \frac{|\partial S|}{|S|} : S \subseteq V(G),\ 0 < |S| \leq n/2 \right\}$$

<u>Trivial</u>: • $h(G) > 0 \iff G$ is connected



The Cheeger constant $h(G)$ measures how small the "bottleneck" in G is.



For any G w/ $0 = \lambda_1 \le \lambda_2 \le \cdots \le \lambda_n$ eig Lap...

Thm: $h(G) \geq \lambda_2 / 2$

1st **Pf**

• Suffices to show that

$$\forall\, S \subseteq V(G) \text{ of size } sn, \quad n = |V(G)|$$

$$\theta(S) \geq (1-s)\lambda_2 .$$

• By Courant-Fischer,

$$\lambda_2 = \min_{\substack{x \neq 0 \\ x \perp 1}} R_L(x) = \min_{\substack{x \neq 0 \\ x \perp 1}} \frac{x^T L x}{x^T x}$$

$$\Rightarrow \forall x \perp 1,\ x \neq 0, \quad \frac{x^T L x}{x^T x} \geq \lambda_2 \quad \cdots (*)$$

We shall use 1_S, but first we center it to be $\perp 1$.

Plug $x = 1_S - s1$, $\langle x, 1 \rangle = 0$ in $(*)$

As $L \cdot 1 = 0$, $x^T L x = 1_S^T L 1_S = \overbrace{|\partial S|}^{= e(S, S^c)}$

$$\Rightarrow \frac{x^T L x}{x^T x} = \frac{|\partial S|}{\underbrace{s(1-s)n}_{= |S| \cdot (1-s)}} \geq \lambda_2 \quad \cdots \Rightarrow \quad \blacksquare$$

$\boxed{2^{nd}\ \text{pf}}$ $h(G) \geq \lambda_2/2$

WTS: $\forall |S| \leq n/2$, $\theta(S) \geq \lambda_2/2$

By C-F: $\lambda_2 = \min_{\dim U = 2} \max_{\substack{x \neq 0 \\ x \in U}} R_L(x)$ $(\heartsuit)$

Take U = span of 1_S & 1_{S^c}

Since $1_S \perp 1_{S^c}$, $\dim U = 2$

Supp. $z \neq 0$, $z \in U$ max: $\max_{\substack{x \neq 0 \\ x \in U}} R_L(x) = R_L(z)$

then $(\heartsuit) \Rightarrow \lambda_2 \leq R_L(z)$

Lem Let M be a positive semidefinite matrix and x, y be two orthogonal vectors.

Then $R_M(x+y) \leq 2 \cdot \max\{R_M(x), R_M(y)\}$.

Recall L is positive semidefinite

Assuming this lemma, write $z = \alpha x + \beta y$, where

$\alpha, \beta \in \mathbb{R}$ and $x = 1_S$, $y = 1_{S^c}$

$\Rightarrow \lambda_2 \leq R_L(z) = R_L(\alpha x + \beta y)$

$\overset{\text{Lemma}}{\leq} 2 \max\{R_L(\alpha x), R_L(\beta y)\}$

Scalar does not affect Rayley quotient! $R_L(\alpha x) = R_L(x)$.

$$= 2 \cdot \max \{ R_L(x), R_L(y) \}$$

$\uparrow$ 1_S $\quad R_L(1_S) = \theta(S)$

$$= 2\theta(S)$$

$\blacksquare$

[Pf of the Lemma] Let $0 \le \lambda_1 \le \lambda_2 \le \cdots \le \lambda_n$ be eigenvalues of M w./ corresp. eig-vector $v_1, \ldots, v_n$ which are basis.

Write $\quad x = \sum_{i=1}^{n} a_i v_i \quad , \quad y = \sum_{i=1}^{n} b_i v_i$

$$R_M(x+y) = \frac{\left(\sum (a_i+b_i) v_i\right)^T M \left(\sum (a_i+b_i) v_i\right)}{\|x+y\|^2}$$

$$= \frac{\sum_{i=1}^{n} \lambda_i (a_i+b_i)^2}{\|x+y\|^2}$$

$(a_i+b_i)^2 \rightsquigarrow \le 2(a_i^2+b_i^2)$

$x \perp y$

$\|x+y\|^2 \rightsquigarrow \|x+y\|^2 = \|x\|^2 + \|y\|^2$

$\sum \lambda_i a_i^2 = x^T M x = R_M(x) \cdot \|x\|^2$

$$\le \frac{\sum \lambda_i \cdot 2(a_i^2 + b_i^2)}{\|x\|^2 + \|y\|^2}$$

$$= \frac{2 R_M(x) \cdot \|x\|^2 + 2 R_M(y) \cdot \|y\|^2}{\|x\|^2 + \|y\|^2}$$

$$\le 2 \cdot \max \{ R_M(x), R_M(y) \}. \quad \blacksquare$$

- Cheeger constant $h(G)$ is often useful when dealing w/ vertex subsets.

- For edge expansion, it's more convenient to work w/ a notion called conductance.

For this, we need to define normalised Laplacian for general graphs.

Recall for d-reg graphs

$$L = dI - A = D - A$$

$$N = I - \frac{1}{d}A$$

It turns out for general graphs

$$\left[L = D - A, \text{ where } D = \begin{pmatrix} d(v_1) & & 0 \\ & \ddots & \\ 0 & & d(v_n) \end{pmatrix} \right]$$

$$N = D^{-1/2} L D^{-1/2}, \text{ where } D^{-1/2} = \begin{pmatrix} d(v_1)^{-1/2} & & \\ & \ddots & \\ & & d(v_n)^{-1/2} \end{pmatrix}$$

$$= D^{-1/2}(D - A)D^{-1/2}$$

$$= I - D^{-1/2} A D^{-1/2}.$$

Next time explain why this defn.