Lecture 23

Recall

- Laplacian: real symm singular & positive semidefinite

  - $L = D - A$

$$\begin{pmatrix} d(v_1) & & -1 \\ & d(v_2) & \\ -1 & & \ddots \; d(v_n) \end{pmatrix}$$

  - $L \cdot \mathbb{1} = 0$

d-reg

$L = d \cdot I - A$

$0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$

$N = I - \frac{1}{d} A$

$0 = \nu_1 \leq \nu_2 \leq \cdots \leq \nu_n$

- Godsil-Newman (Hoffman's bdd for irreg)

$$\underbrace{|I|}_{\text{indep set}} \leq \frac{\lambda_n - \underbrace{d(I)}_{\text{ave deg in } I}}{\lambda_n} \cdot n$$

pf: $\forall x \neq 0$, $\lambda_n \geq R_L(x) = \dfrac{x^T L x}{x^T x}$

$x = \mathbb{1}_I - \frac{|I|}{n} \cdot \mathbb{1}$, $x \perp 1$ ▨

- quadratic $\quad x^T L x = \sum_{uv \in E} (x_u - x_v)^2 \geq 0$

cut if $x \in \{0,1\}^V$

- Operator $\quad (Lx)_v = d\left(x_v - \frac{1}{d} \sum_{u \in N(v)} x_u\right)$

Scaled diff. btw x_v & average of $N(v)$

d-reg

- Consider an n-vx $G=(V,E)$ w./ normalised Laplacian $N = I - \frac{1}{d}A$.

$$\begin{pmatrix} 1 & & -\frac{1}{d} \\ & 1 & \\ & & 1 & 0 \\ -\frac{1}{d} & & & 1 \end{pmatrix}$$

$$x^T N x = \frac{1}{d}\left(\sum_{v\in V} d x_v^2 - \cancel{\frac{1}{d}} \sum_{uv\in E} 2x_u x_v\right)$$

$$= \sum_{u,v\in V} x_u x_v N_{uv}$$

$$= \frac{1}{d}\sum_{uv\in E}(x_u - x_v)^2$$

$$R_N(x) = \frac{\sum_{uv\in E}(x_u-x_v)^2}{d\sum_{v\in V} x_v^2}$$

<u>Thm</u>: Let $G=(V,E)$ be n-vx d-reg graph and let $N = I - \frac{1}{d}A$ be its normalised Laplacian w./ eigenvalue $0 \leq \nu_1 \leq \nu_2 \leq \dots \leq \nu_n \leq 2$

Then (i) $\nu_1 = 0$

(ii) $\nu_k = 0 \iff G$ has $\geq k$ (conn) components

In particular, # comp. in G = multiplicity of the eigenvalue 0.

(iii) $\nu_n \leq 2$ w./ equality $\iff G$ has a bipartite comp.

Pf (i) $\nu_1 = \min_{x \neq 0} R_N(x) = \min_{x \neq 0} \frac{\sum_{uv \in E} (x_u - x_v)^2}{d \sum_{v \in V} x_v^2} \geq 0$

On the other hand, $N \cdot \mathbb{1} = 0$

$\Rightarrow \nu_1 = 0.$

(ii) ($\Rightarrow$) Suppose $\nu_k = 0$, WTS #comp. $\geq k$.

By C-F:

$$\nu_k = \min_{\dim U = k} \max_{\substack{x \in U \\ x \neq 0}} R_N(x)$$

$$= \min_{\dim U = k} \max_{\substack{x \in U \\ x \neq 0}} \frac{\sum_{uv \in E} (x_u - x_v)^2}{d \sum_{v \in V} x_v^2}$$

$$= 0$$

Thus, $\exists$ a k-dim sp. U s.t. $\forall x \in U$, $x \neq 0$

we have $R_N(x) = \frac{\sum (x_u - x_v)^2}{d \sum x_v^2} = 0$

$\Rightarrow$ if $u \sim v \Rightarrow x_u = x_v$

G

$\Rightarrow$ x is constant over any conn. component.

$\Rightarrow k = \dim U \leq$ #comp. of G

$(\Leftarrow)$: Supp. # comp. of $G \geq k$.

Consider the subsp. U spanned by vectors that are const. on each component.

$\Rightarrow$ $\dim U = \#\text{comp.} \geq k$.

In this U, $\forall x \neq 0$, $x \in U$

$$R_N(x) = 0 \xrightarrow{\text{C.F.}} \nu_k = 0.$$

(iii).

$$\nu_n = \max_{x \neq 0} R_N(x) = \max_{x \neq 0} \frac{x^T N x}{x^T x}$$

$$= \max_{x \neq 0} \frac{\sum_{uv \in E} (x_u - x_v)^2}{d \sum_{v \in V} x_v^2}$$

$$x^T N x = \frac{1}{d} \sum_{uv \in E} (x_u - x_v)^2 = \sum_{v \in V} x_v^2 - \frac{2}{d} \sum_{uv \in E} x_u x_v$$

$$= 2x^T x - \frac{2}{d} \sum_{uv \in E} x_u x_v - x^T x$$

$$= 2x^T x - \frac{1}{d} \sum_{uv \in E} (x_u + x_v)^2$$

$$R_N(x) = 2 - \frac{\sum_{uv \in E} (x_u + x_v)^2}{d \sum_{v \in V} x_v^2} \leq 2$$

$\Rightarrow$ $\nu_2 \leq 2$

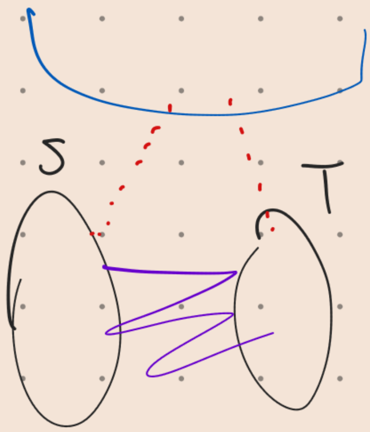
If $\nu_2 = 2$, then we must have

$$\boxed{\sum_{uv \in E} (x_u + x_v)^2 = 0 \qquad (*)}$$

So $u \sim v \Rightarrow x_u = -x_v$

<u>Def</u> $S = \{v : x_v > 0\}$

$T = \{v : x_v < 0\}$



- $S \cup T$ sends no edge to $V \setminus (S \cup T)$, for o.w.

  such edge contributes positively to $\sum_{uv \in E} (x_u + x_v)^2$

  $\lightning$ $(*)$

- Also $N(S) \subseteq T$, $N(T) \subseteq S$ by $(*)$

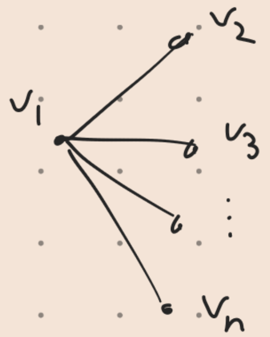
$\Rightarrow$ $S \cup T$ induces union of bipartite comp. in G.

— Examples —

<u>Exer</u>: L_{K_n}

	eigenvalue	eigenvector	multiplicity
L_{K_n}	0	$\mathbb{1}$	1
	n	$\forall x \perp \mathbb{1}$	$n-1$

Consider S_n star



eigenvalue	eigenvector	multiplicity
0	$\mathbb{1}$	1
1	$\delta_{v_i} - \delta_{v_{i+1}}$	$n-2$
n	$\begin{pmatrix} -(n-1) \\ 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = y$	1

$\delta_v = \mathbb{1}_{\{v\}}$

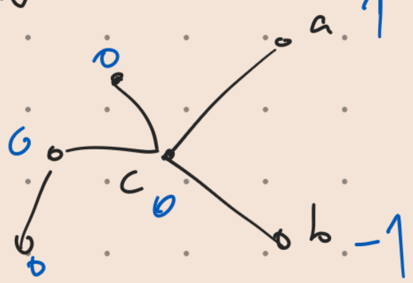
orthog

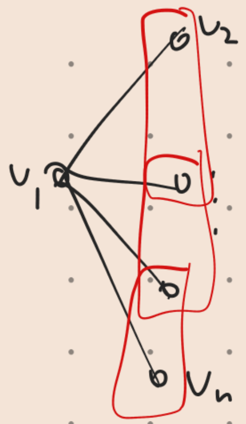
y_{v_i} same for all $2 \le i \le n$

$2 \le i \le n-1$

<u>Lem</u> Let $G = (V, E)$ be an n-vx graph w/ two deg-1 vertices a & b both adj. to another vertex c, then $x = \delta_a - \delta_b$ is an eigenvector of G w/ eigenvalue 1.

<u>Pf</u>: EXERCISE. $(Lx)_v = x_v$





$$\left.\begin{matrix} v_2 - v_3 \\ v_3 - v_4 \\ \vdots \\ v_{n-1} - v_n \end{matrix}\right\} n-2$$

$$L = \begin{pmatrix} n-1 & -1 & & & & \\ & 1 & & & & \\ -1 & & 1 & & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{pmatrix}$$

sum of diag. entries

$$\mathrm{tr}(L) = 2n-2 = \sum_{i=1}^{n} \lambda_i$$

$$= n-2+0+\lambda_n$$

$$\Rightarrow \lambda_n = n$$

Def: $G_1 = (V_1, E_1)$

$G_2 = (V_2, E_2)$

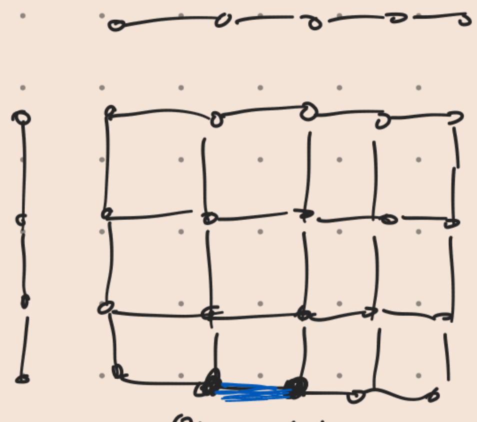
$V(G_1 \times G_2) = V_1 \times V_2$

$E(G_1 \times G_2)$: $(x, y) \sim (x', y)$

where $xx' \in E_1$

$(x, y) \sim (x, y')$

where $yy' \in E_2$.



Thm: Given $G_i = (V_i, E_i)$, $i \in [2]$, w/ Laplacian

eigenvalues $\lambda_1^{(1)} \le \cdots \le \lambda_n^{(1)}$ & $\lambda_1^{(2)} \le \cdots \le \lambda_m^{(2)}$

eigenvector $x_1, \ldots, x_n$ $\quad y_1, \ldots, y_m$

Then $G_1 \times G_2$ has

eigenvalues $\lambda_i^{(1)} + \lambda_j^{(2)}$ $\quad \forall\, i \in [n]$, $\forall\, j \in [m]$ w/

eigenvector $z_{i,j}$: $(z_{i,j})_{(u,v)} = (x_i)_u \cdot (y_j)_v$.

- d-dim Hypercube Q_d $\quad L_{Q_d}$

 Q_1

	eigenvalue	eigenvector
	0	$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$
	2	$\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$$\begin{array}{ccc} Q_{d-1} & \lambda & x \\ & \Downarrow & \\ Q_d & \lambda & \begin{pmatrix} x \\ x \end{pmatrix} \\ & \lambda+2 & \begin{pmatrix} x \\ -x \end{pmatrix} \end{array} \Longrightarrow$$

Q_d has eigenvalues $2i$ for each $0 \le i \le d$ w/ multiplicity $\binom{d}{i}$.

We can identify eigenvectors of L_{Q_d} w/ $V(Q_d)$

Take $v \in \{0,1\}^d = V(Q_d)$

$$\Downarrow$$

eigenvector $x^{(v)}$: $\forall u \in V(Q_d)$

$$x^{(v)}_u = (-1)^{v^T u}.$$