Lecture 22

Recall:

- Spectral thm: real symm $\Rightarrow$ real eigenvalues & real orthogonal eigenvectors

  $\forall x \in \mathbb{R}^V$, $x = \sum a_i v_i$ ← adj matrix of G

  $a_1, \sum a_i^2$ linked to $|X|$ if $x = \mathbb{1}_X$

- operator $x \longmapsto Mx$

  $(Ax)_v$ = sum of $N(v)$ $= \sum_{u \in N(v)} x_u$

  $x^T M x$, $x^T A x = 2e(A)$

- Hoffman: $\alpha(G) \leq \dfrac{-\alpha_n}{d - \alpha_n} \cdot n$ $\forall$ n-vx d-reg G

  A adj matrix, $\alpha_1 = d$, $\alpha_2, \ldots, \alpha_n$ eigenvalues of A.

  $$\chi(G) \geq \frac{n}{\alpha(G)} = \frac{d - \alpha_n}{-\alpha_n}$$

- Rayleigh quotient $R_M(x) = \dfrac{x^T M x}{x^T x}$

  Courant-Fischer M real symm

  $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$

  $$\lambda_k = \min_{k\text{-dim } U} \max_{\substack{x \in U \\ x \neq 0}} R_M(x) = \min_{\ldots} \max_{\ldots} \frac{x^T M x}{x^T x}.$$

C-F

Rmk: $\Rightarrow$ View eigenvalues as optima of min-max optimisation problem in which the cost funct is the Rayleigh quotient.

- Usually important ones are λ_1, λ_2 & λ_n.

Exer: $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$

Cor:
$$\lambda_1 = \min_{x \neq 0} R_M(x) = \min_{x \neq 0} \frac{x^T M x}{x^T x}$$

$$\lambda_n = \max_{x \neq 0} R_M(x) = \max_{x \neq 0} \frac{x^T M x}{x^T x}$$

$$\lambda_2 = \min_{x \neq 0, x \perp v_1} R_M(x) = \min_{\dots} \frac{x^T M x}{x^T x}$$

eigenvector of λ_1

§ Laplacian

Consider first n-vx d-reg G

Def: $L = d \cdot I - A = \begin{array}{c|c} & v \\ \hline u & \begin{pmatrix} d & & & \\ & d & & -1 \\ & & \ddots & \\ & & & d \end{pmatrix} \end{array}$ $u \sim v$

Facts:
- $L \cdot \mathbb{1} = 0$
- v_i is an eigenvector of A w/ eigenvalue α_i

$\Rightarrow$ v_i is also ...... L ... ... $d - \alpha_i$

- Normalised Laplacian for d-reg

$$N = \frac{1}{d} \cdot L = I - \frac{1}{d} \cdot A \quad \left(\text{spectral radius indep of } d \right)$$

- For general graph

$$L = D - A, \text{ where } D = \begin{pmatrix} d(v_1) & & 0 \\ & d(v_2) & \\ 0 & & \ddots \; d(v_n) \end{pmatrix}$$

<u>Convention</u>: A adj matrix

$\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n$

L Laplacian

$\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$

N Normalised Laplacian

$\nu_1 \leq \nu_2 \leq \cdots \leq \nu_n$.

§ Hoffman's bound for irreg graphs

<u>Thm</u> [Godsil - Newman]

Let I be an indep set in G and let $d(I)$ be the ave. deg of vertices in I.

$$\Rightarrow \quad |I| \leq \frac{\lambda_n - d(I)}{\lambda_n} \cdot n,$$

where $\lambda_1 \leq \cdots \leq \lambda_n$ are eigenvalues of Laplacian of G.

Rmk: If G is d-reg, then it implies Hoffman's bdd.

$\lambda_n = d - \alpha_n$, α_i: eigenvalues of A.

$$|I| \le \frac{\lambda_n - d}{d - \alpha_n} \cdot n = \frac{-\alpha_n}{d - \alpha_n} \cdot n.$$

Pf: By Courant-Fischer

$$\lambda_n = \max_{x \neq 0} R_L(x) = \max_{x \neq 0} \frac{x^T L x}{x^T x}$$

$$\Rightarrow \forall x \neq 0, \quad \boxed{\lambda_n \geq R_L(x) = \frac{x^T L x}{x^T x}}$$

Let $\underbrace{x = \mathbb{1}_I - \frac{|I|}{n} \cdot \mathbb{1}}$ $= \begin{pmatrix} \overset{\notin I}{-\frac{|I|}{n}}, & \overset{\in I}{1 - \frac{|I|}{n}}, & \cdots \end{pmatrix}$

Rmk: The reason to choose centered characteristic funct. of I instead of $\mathbb{1}_I$ is because $x^T L x$ remain the same, $L \cdot \mathbb{1} = 0$ shifting by $\frac{|I|}{n}\mathbb{1}$ minimises the norm $x^T x$.

So $x \perp \mathbb{1}$

- $$x^T L x = \left(\mathbb{1}_I - \frac{|I|}{n}\mathbb{1}\right)^T L \left(\mathbb{1}_I - \frac{|I|}{n}\mathbb{1}\right)$$

as $L \cdot \mathbb{1} = 0$
$$= \mathbb{1}_I^T L \, \mathbb{1}_I = \mathbb{1}_I^T (D - A) \mathbb{1}_I$$

$$= \sum_{u,v} (\mathbb{1}_I)_u \cdot (D-A)_{u,v} \cdot (\mathbb{1}_I)_v$$

$$= \sum_{v \in I} d(v) - \cancel{2e(I)}^{\,0}$$

$$= |I| \cdot d(I)$$

$$x^T x = \left(\mathbb{1}_I - \frac{|I|}{n}\mathbb{1}\right)^T \left(\mathbb{1}_I - \frac{|I|}{n}\mathbb{1}\right) \overset{ex}{=} |I|\left(1 - \frac{|I|}{n}\right)$$

$$\Rightarrow \lambda_n \geq \frac{\cancel{|I|} \cdot d(I)}{\cancel{|I|}\left(1 - \frac{|I|}{n}\right)} \Rightarrow$$

<u>Ex</u>: Let $S \subseteq V$ be a set of size $s \cdot |V|$

Let $f_t = \mathbb{1}_S - t \cdot \mathbb{1}$, then

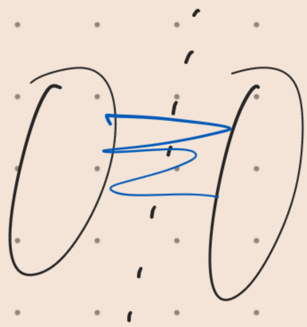
$\|f_t\|^2$ is minimised when $t = s$, i.e.

when $f_t \perp \mathbb{1}$, and

$\|\mathbb{1}_S - s \cdot \mathbb{1}\|^2 = s(1-s)|V|$.

— § Why this defn. of Laplacian —

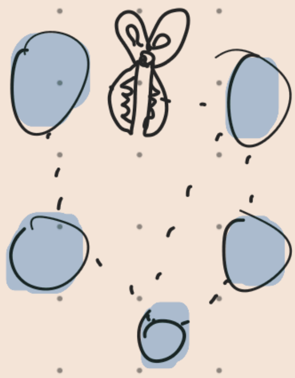
<u>Motivation</u>: G



<u>Cut problem</u>: <u>ex</u>: $\exists$ a cut (always) w/ $\geq \frac{1}{2} e(G)$

X $X^c = V \setminus X$

· TCS: finding sparse cut $\longrightarrow$ divide & conquer



An important deg-2 homomogeneous polyn. for graphs:

$G = (V, E)$

$$\sum_{uv \in E} (x_u - x_v)^2$$

- measures smoothness of x: small when no big jump over edges.
- When $x \in \{0,1\}^V$ is Boolean vector. $x = 1_X$

$$e(X, X^c) = \sum_{uv \in E} (x_u - x_v)^2$$

Every homog. quadratic polyn. can be written as $x^T M x$ for some matrix M.

d-reg graph

$$\sum_{uv \in E} (x_u - x_v)^2 = x^T (dI - A) x = x^T L x$$

both sides $\stackrel{ex}{=} \sum_{v \in V} d\, x_v^2 - 2 \sum_{uv \in E} x_u x_v$.

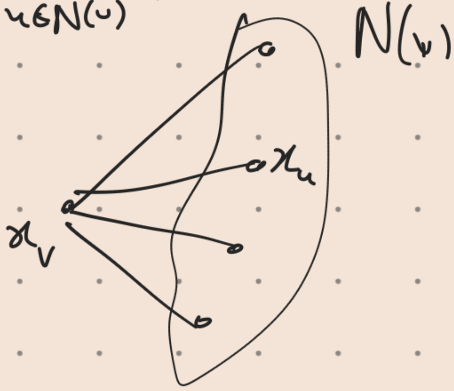
<u>Prop</u>: The Laplacian L is singular & positive semidefinite.

What does L do?

- operator $x \in \mathbb{R}^V$

$$(Lx)_v = \begin{array}{c} \\ \\ v \\ \\ \end{array}\begin{pmatrix} d & & & \\ & \ddots & & -1 \\ & & d & \\ & & & d \end{pmatrix} \overset{x}{\begin{pmatrix} \vdots \\ \vdots \\ 1 \end{pmatrix}} = d x_v - \sum_{u \in N(v)} x_u$$

$$= d\left(x_v - \frac{1}{d} \sum_{u \in N(v)} x_u\right)$$



- quadratic

$$x^T L x = \sum_{uv \in E} (x_u - x_v)^2.$$

- cut value when $x \in \{0,1\}^V$.

$x \in \mathbb{R}^V$, optimising $x^T L x$ can be viewed as a relaxation of the cut problem.