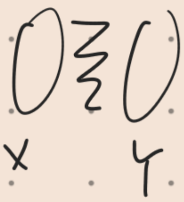
Lecture 21

Spectral methods use linear algebra to derive information about graphs, usually via studying certain (often symmetric) real matrices associated to graphs (adj / Laplacian / transition etc)

- $G = (V, E)$, $|V| = n$
  - Set of vxs $X \subseteq V$ $\rightsquigarrow$ 0,1-vector in $\{0,1\}^V$
    Cut of a graph (X, X^c)
  - $\mathbb{R}^V$ $\rightsquigarrow$ fractional cut
    prob. distribution over V (when normalised)
- edge sets $\rightsquigarrow$ quadratic forms defined by it

$$X^T A Y = e(X, Y)$$
$$= \sum_{u,v \in V} X_u A_{uv} Y_v$$



When dealing with matrices M, we usually use them in two ways:

as (i) operator $x \in \mathbb{R}^V \longmapsto Mx \in \mathbb{R}^V$

(ii) define a quadratic form $x \longmapsto x^T M x \in \mathbb{R}$

Thm (Spectral thm)

Let $M \in \mathbb{R}^{n \times n}$ be a real symmetric $n \times n$ matrix. Then it has n real eigenvalues (not necessarily distinct) $\lambda_1, \dots, \lambda_n$ and n orthonormal real vectors $x_1, \dots, x_n \in \mathbb{R}^n$ where x_i is an eigenvector of λ_i.

- Shall A for adj matrix of G

$$A_{uv} = 1 \iff u \sim v \qquad \begin{array}{cc} & \begin{array}{cc} u & v \end{array} \\ u & \begin{pmatrix} 0 & & & & \\ & 0 & & & \\ & & 0 & & \\ & & & 1\,0\,0 & \\ & & & 0 & \\ & & & & 0\,0 \end{pmatrix} \end{array}$$

If G is n-vx d-regular

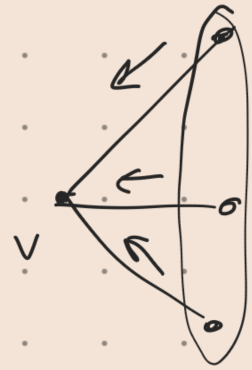
Basics :
- $\mathbb{1}$ is an eigenvector for adj matrix A w/ eigenvalue d.
- $d = \lambda_1 \geq \cdots \geq \lambda_n$

  $\lambda_n < 0$, $tr(A) = 0 = \sum \lambda_n$
- multiplicity of $\lambda_1 = d$ is # conn. components in G.

- For adj matrix A
  - as an operator: take $x \in \mathbb{R}^V$

$$Ax = \begin{array}{cc} & u \in N(v) \\ v & \begin{pmatrix} \\ \cdots 1 \cdots 1 \cdots 0 \cdots \\ \end{pmatrix} \end{array} \begin{pmatrix} x_{v_1} \\ x_{v_2} \\ \vdots \\ x_{v_n} \end{pmatrix}$$

$$(Ax)_v = \sum_{u \in N(v)} x_u$$



$(Ax)_v$ is the sum of weights at $N(v)$

- $x^T A x = e(X,X) = 2e(X)$

<u>Ex</u> Complete graph K_n

eigenvalue	multiplicity	
$n-1$	1	$\leftarrow$ exercise
-1	$n-1$	

$\mathbb{1}$ eigenvector for $n-1$, take $x \perp \mathbb{1}$, $\langle x, \mathbb{1} \rangle = \sum x_v = 0$

$$(Ax)_v = \underbrace{\sum_{u \in N(v)}}_{V \setminus \{v\}} x_u = -x_v$$

$\langle x, 1 \rangle = 0$

$$\Rightarrow \quad Ax = -x \quad \forall\, x \perp 1 \quad \square$$

Hoffman's bound

<u>Thm</u> $\forall$ n-vtx d-regular graph G. Let $d \geq \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ be eigenvalues of its adj matrix.

Then
$$\alpha(G) \leq \frac{-\lambda_n}{d-\lambda_n} \cdot n$$

<u>Exam</u> $K_{d,d}$ $\quad \lambda_1 = d$, $\lambda_n = -d$.

- By spectral thm, we have $v_1, \dots, v_n$ orthonormal eigenvectors for $\overset{\text{adj matrix}}{A}$ for n-vx d-regular G.

$\forall$ set of vxs $X \subseteq V$,

$$X = \sum_{i \in V} a_i v_i$$

$$v_1 = \frac{1}{\sqrt{n}} \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$$

- a_1 meaning: $\langle X, v_1 \rangle = \langle \sum a_i v_i, v_1 \rangle \overset{\text{orthog}}{=} a_1$

$$\langle X, v_1 \rangle = \langle X, \tfrac{1}{\sqrt{n}} \mathbf{1} \rangle = \frac{1}{\sqrt{n}} |X|.$$

- $\sum a_i^2$: $\langle X, X \rangle = \langle \sum a_i v_i, \sum a_i v_i \rangle = \sum a_i^2$

$$\langle X, X \rangle = |X|$$

<u>pf</u>: (Hoffman's bound) Take an indep set I in G

$$I = \sum a_i v_i \qquad \begin{cases} a_1 = \frac{1}{\sqrt{n}} |I| \\ \sum a_i^2 = |I| \end{cases} \qquad a_1^2 = \frac{1}{n} |I|^2$$

$$0 \overset{\text{indep set}}{=} 2e(I) = I^T A I = \langle I, AI \rangle = \langle \sum a_i v_i, \sum a_i A v_i \rangle$$

$$= \langle \sum a_i v_i, \sum \lambda_i a_i v_i \rangle$$

$$\overset{v_i \perp v_j}{=} \sum \lambda_i a_i^2$$

$$= d a_1^2 + \sum_{i \geq 2} \lambda_i a_i^2$$

$$\geq d a_1^2 + \sum_{i \geq 2} \lambda_n a_i^2$$

$$= (d - \lambda_n) a_1^2 + \left(\sum_{i=1}^{n} a_i^2 \right) \cdot \lambda_n$$

$$\Longleftrightarrow \quad 0 \geq (d - \lambda_n) \frac{|I|^2}{n} + |I| \cdot \lambda_n$$

Solve it $\longrightarrow$ 

<u>Cor</u> $\forall$ n-vx d-reg graph, $\lambda_1 \geq \cdots \geq \lambda_n$ adj mat.

$$\chi(G) \geq \frac{n}{\alpha(G)} \geq \frac{\lambda_1 - \lambda_n}{-\lambda_n}$$

<u>Rmk</u>. $\chi(G) \geq \dfrac{\lambda_1 - \lambda_n}{-\lambda_n}$ also holds for irregular graphs.

<u>Def</u> The Rayleigh quotient of x w/ respect to M is

$$\frac{x^T M x}{x^T x} = R_M(x)$$

<u>Fact</u>: If x is an eigenvector of M w/ eigenvalue λ.

then the Rayleigh quotient $R_M(x) = \lambda$

$$\frac{x^T M x}{x^T x} = \frac{x^T \lambda x}{x^T x} = \lambda.$$

- The variational characterisation of eigenvalues.

<u>Thm</u> [Courant-Fischer]

Let $M \in \mathbb{R}^{n\times n}$ a real $n\times n$ symmetric matrix,

and $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$ be its eigenvalues.

Then

$$\lambda_k = \min_{\substack{k\text{-dim}\\ V}} \ \max_{\substack{x\in V\\ x\neq 0}} \frac{x^T M x}{x^T x} = \min_{\substack{k\text{-dim}\\ V}} \ \max_{\substack{x\in V\\ x\neq 0}} R_M(x)$$

<u>Pf</u>: • By spectral thm, we can take

orthonormal eigenvectors

$n-k+1$

$$v_1, v_2, \dots, v_{k-1}, v_k, \dots, v_n$$

$$\lambda_1 \leq \lambda_2 \leq \dots \leq \dots \leq \lambda_n$$

- "$\geq$" Need to find some k-dim V s.t.

$$\lambda_k \geq \max_{\substack{x\in V\\ x\neq 0}} R_M(x)$$

Consider $V =$ span of $v_1, \dots, v_k$

and take non-zero $x \in V$,

$$x = \sum_{i=1}^{k} a_i v_i$$

$$x^T M x = x^T M\left(\sum_{i=1}^{k} a_i \cdot v_i\right)$$

$$= \left\langle \sum_{i=1}^{k} a_i v_i , \sum_{i=1}^{k} a_i \lambda_i v_i \right\rangle$$

$$\overset{\perp}{=} \sum_{i=1}^{k} \lambda_i a_i^2 \qquad \lambda_k \geq R_M(x)$$

$$\leq \lambda_k \cdot \sum_{i=1}^{k} a_i^2$$

$$x^T x = \left\langle \sum_{i=1}^{k} a_i v_i , \sum a_i v_i \right\rangle \overset{\perp}{=} \sum_{i=1}^{k} a_i^2 .$$

- " $\leq$ " $\quad \lambda_k \leq \min_{k\text{-dim } V} \max_{\substack{x \in V \\ x \neq 0}} \frac{x^T M x}{x^T x}$

Need to show that $\forall$ k-dim V, we can find a nonzero $x \in V$ w/ $R_M(x) \geq \lambda_k$.

- Fix an arbitrary k-dim sp. V.

and let U = span of $v_k, v_{k+1}, \dots, v_n$

$\dim U = n-k+1$

As $\dim U + \dim V > n$

$\Rightarrow \exists$ non zero $x \in U \cap V$

Write $x = \sum_{i=k}^{n} a_i v_i \qquad x^T x \overset{\perp}{=} \sum_{i=k}^{n} a_i^2$

$$x^T M x = \langle \sum a_i v_i , \sum a_i \lambda_i v_i \rangle = \sum_{i=k}^{n} \lambda_i a_i^2 \geq \lambda_k \sum_{i=k}^{n} a_i^2 \quad \blacksquare$$