Lecture 20

Thm Let G be a bipartite graph w/ bipartition X and Y, each of size N. Suppose G has pN^2 edges. Then the following properties are equivalent.

(i) (C_4-count) # labeled 4-cycles in G is at most $p^4N^4 + o(N^4)$

(ii) (Discrepancy) $\forall\ X' \subseteq X,\ Y' \subseteq Y$,
$$\big|e(X',Y') - p|X'||Y'|\big| = o(N^2)$$

regular pair defn in Szemerédi's regularity lemma

Thm (Chung-Graham 92)

Let A be a subset of $\mathbb{Z}_N$ of size $p \cdot N$. Then the following properties are equivalent.

(i) # quadruples $(a,b,c,d) \in A^4$ s.t.

Sidon $\rightarrow$ $a+b=c+d$ is at most $p^4N^3 + o(N^3)$

(ii) $\forall$ arith. prog. X in $\mathbb{Z}_N$

$$\Rightarrow \quad |A \cap X| = p|X| + o(N)$$

- Connection btw quasirandom sets in $\mathbb{Z}_N$ & quasirandom graphs.

Take $A \subseteq \mathbb{Z}_N$, $|A| = pN$

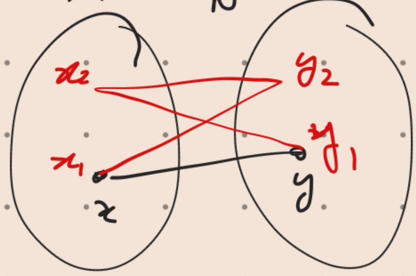
Consider the following Cayley sum graph

$X = \mathbb{Z}_N = Y$

$V(G) = X \cup Y$

$x \sim y \iff x + y \in A$

$e(G) = pN^2$



A quasirandom set $\iff$ G quasirandom graph

- Take a C_4 in G, say $x_1 y_1 x_2 y_2$

$\Rightarrow$ $x_1 + y_1$, $x_1 + y_2$, $x_2 + y_1$, $x_2 + y_2 \in A$

$a + b = c + d$

$(x_1 + y_1) + (x_2 + y_2) = (x_1 + y_2) + (x_2 + y_1)$

N choices for x_1

N-to-1 correspondence btw 4-cycles & $a+b=c+d$

$p^4 N^4$ $\qquad$ $p^4 N^3$

- What about quasirandom hypergraphs, how to define them?

Consider 3-unif hypergraphs.

When a set is quasirandom, # 3AP counts is as expected.

[# sol^ns to $a+b=c+d \leq p^4 N^3 + o(N^3)$]

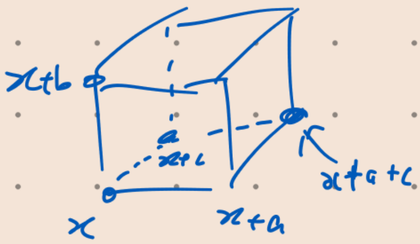
But there are quasirandom set whose 4-AP count deviates a lot from expected.

To detect whether a set has abnormal count of 4-APs, Gowers introduced a high order quasirandomness.

<u>Def</u> (quadratic uniformity) Let $A \subseteq \mathbb{Z}_N$ of size pN.

We say A is quadratically uniform if

\# octuples $(x, x+a, x+b, x+c, x+a+b, x+a+c, x+b+c, x+a+b+c)$ in A^8 is at most $p^4 N^4 + o(N^4)$.

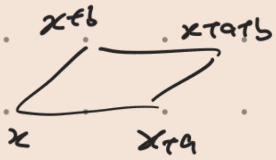


<u>Rmk</u>: Natural generalisation:

$$a+b = c+d$$

$$\Updownarrow$$

$$x, x+a, x+b, x+a+b$$

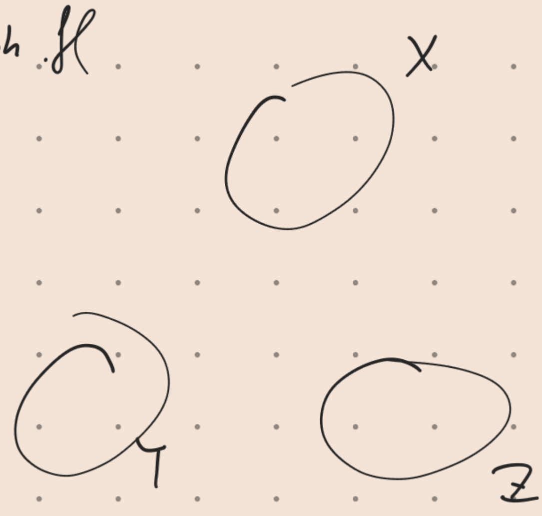


- To define quasirandom properties for 3-unif hypergraphs.
  - $A \subseteq \mathbb{Z}_N$

Consider Cayley sum hypergraph $\mathcal{H}$

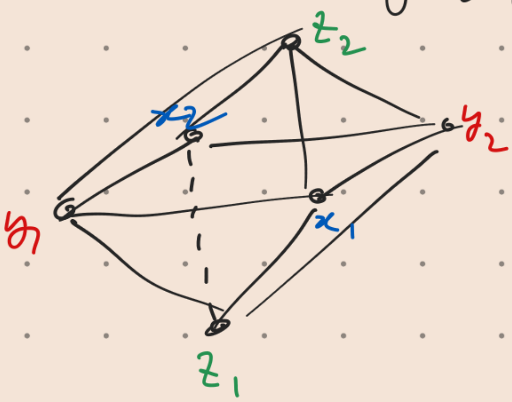
- $X = Y = Z = \mathbb{Z}_N$
- $(x, y, z) \in \mathcal{H}$

  $\Leftrightarrow x+y+z \in A$

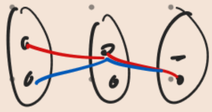


It turns out a counterpart of C_4 in 3-unif hypergraph is Octahedron

6-vx 8-edge 3-graph.



$K^{(3)}_{2,2,2}$



x_i y_i z_i

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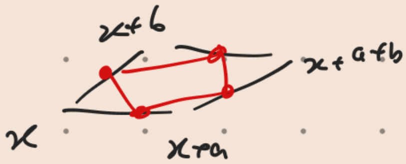
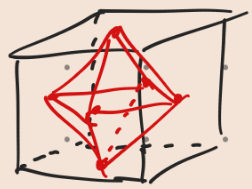
graphs



3-unif hypergraphs.



Dual: 2-dim cube / 3-dim cube

4-cycle / Octahedron

Def. (Octahedron count) Let H be a tripartite 3-unif hypergraph w./ N vxs in each partite set and pN^3 edges.

Then H is quasirandom if it contains

$$\le p^8 N^6 + o(N^6)$$

Rmk: $A \subseteq \mathbb{Z}_N$ quadratically uniform $\Leftrightarrow$ Cayley sum hypeg. is quasirandom

- What about [Disc] for 3-unif hypergraphs?

A natural way is the following:

Def (vertex-uniformity) Let H be a 3-unif tripartite hyperg. w/ partite sets X, Y, Z each of size N and H has pN^3 edges.

Then H is vertex-uniform if

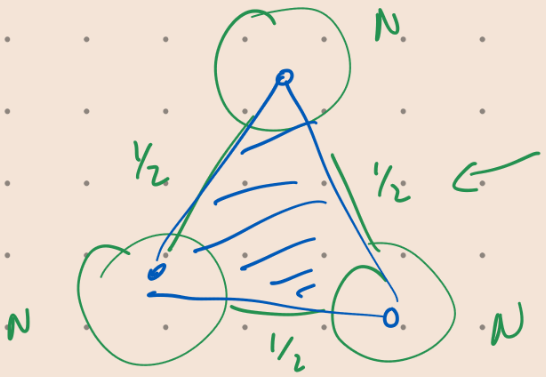
$$\forall X' \subseteq X,\ Y' \subseteq Y,\ Z' \subseteq Z$$

$$\left| e(X', Y', Z') - p|X'||Y'||Z'| \right| = o(N^3).$$

Problem! quasirandom 3-graphs $\Rightarrow$ vertex-uniform
(correct octahedron count)

$\not\Leftarrow$ NOT necessarily.

Ex.

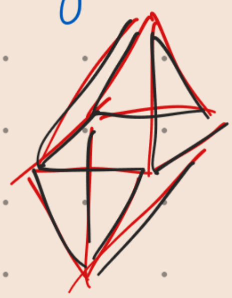


$\leftarrow$ random graph w/ density $\frac{1}{2}$

every $\Delta \Rightarrow$ hyperedge H

- edge density of H is $\frac{1}{8}$
- H is vertex-unif., but not having the right octahedron count:

$$\mathbb{E}\ \#\text{octahedron} = \left(\frac{1}{2}\right)^{12} N^6$$



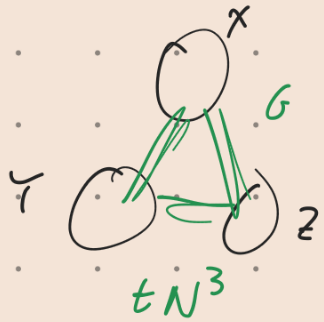
But correct count if quasirandom is $\left(\frac{1}{8}\right)^8 N^6$.

(edge-uniformity)

<u>Def</u>: Let H be a 3-unif tripartite hyperg. w/ X, Y, Z N vxs in each partite set and has pN^3 edges.

Then H is edge-uniform if $\forall\ t \in [0,1]$ and $\forall$ tripartite graph G w/ vertex sets X, Y, Z and tN^3 triangles,

Δs in G that are edges in H is $ptN^3 \pm o(N^3)$.



- The discrepancy property for graphs basically says that a bipartite graph does not significantly correlate w/ graphs induced by set of vertices.

- Edge-uniformity says that 3-unif hyperg. does not significantly correlate w/ 3-graphs induced by set of edges.

(Gowers) Quasirandomness, counting and Regularity for 3-uniform hypergraphs.