Lecture 2 Sampling with Hard-core model.

Last time:

- $\forall$ n-vx Δ-free w./ $\Delta(G) \leq d$

$$\Rightarrow \quad \bar{\alpha}(G) \geq (1+o(1)) \frac{\log_2 d}{4\, d} \cdot n$$

$$\Omega\left(\frac{\log d}{d} \cdot n\right)$$

Conj 81' Ajtai-Erdős-Komlós-Szemerédi:

$\forall$ n-vx K_t-free w./ $\Delta(G) \leq d$

$$\Rightarrow \quad \alpha(G) \geq \Omega\left(\frac{\log d}{d} \cdot n\right)$$

Thm (Shearer 90s) ..... $\Rightarrow \alpha(G) \geq \Omega\left(\dfrac{\log d}{d \cdot \log\log d} \cdot n\right)$

- Similar ($\log d$)-factor improvement is also true for graphs w./ few Δs (locally sparse).

Thm $(\cdots)$ n-vx G w./ $\Delta(G) = d$

Suppose $\#\Delta$s in G is $\leq d^{2-\varepsilon} \cdot n$

$$\Rightarrow \quad \alpha(G) \geq \Omega\left(\frac{\log d}{d} \cdot n\right)$$

(Shearer / Davis-Jenssen, Perkins, Roberts 17)

Thm n-vx Δ-free G w/ $\Delta(G) \le d$

$$\Rightarrow \quad \bar{\alpha}(G) \geqslant (1+o(1)) \frac{\log d}{d} \cdot n$$

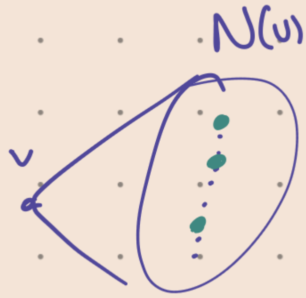
Review the strategy

Today: sample differently

- Sample I uniform random indep set

  Goal: $\mathbb{E}|I| = \bar{\alpha}(G) \geqslant \cdots\cdots$

- Count in two ways



$$(*) \quad \mathbb{E}|I| = \sum_{v \in V(G)} \Pr(v \in I) \quad \cdots\cdots \quad ①$$

$$\geqslant \frac{1}{d} \sum_{v \in V(G)} \sum_{u \in N(v)} \Pr(u \in I) \quad \cdots\cdots \quad ②$$

$$= \sum_{v \in V(G)} \frac{1}{d} \mathbb{E}\left| N(v) \cap I \right|$$

- Sampling via Hard-core model. (Stat. phy.)

$$\Pr(I) \propto \lambda^{|I|}, \quad \lambda > 0: \text{fugacity}.$$

Def: (normalizing factor) Partition function

$$P_G(\lambda) = \sum_{J \in \mathcal{I}(G)} \lambda^{|J|}$$

In Hard-core model, sample indep set:

$$\Pr(I) = \frac{\lambda^{|I|}}{P_G(\lambda)} = \frac{\lambda^{|I|}}{\sum_{J \in \mathcal{I}(G)} \lambda^{|J|}}$$

Rmk: $\lambda = 1$, # indep sets in G $\Rightarrow$ uniform distribution

Def: Let I be sampled $\sim$ Hard-core model w/ fug. λ.

$$\mathbb{E}|I| =: \bar{\alpha}_G(\lambda)$$

Prop $\bar{\alpha}_G(\lambda)$ is the scaled log derative of the partition function

$$\mathbb{E}|I| = \bar{\alpha}_G(\lambda) = \lambda \cdot \left(\log P_G(\lambda)\right)'.$$

Pf: $\bar{\alpha}_G(\lambda) = \mathbb{E}|I| = \sum_{I \in \mathcal{I}(G)} |I| \cdot \Pr(I)$

$$= \sum_I |I| \cdot \frac{\lambda^{|I|}}{P_G(\lambda)} = \lambda \cdot \frac{P_G'(\lambda)}{P_G(\lambda)} = \lambda \left(\log P_G(\lambda)\right)'.$$

- For instance, say $\bar{\alpha}_G(\lambda) \geq \cdots$

$\Rightarrow$ a lower bound on $P_G(\lambda)$.

Setting $\lambda = 1 \Rightarrow$ count # indep sets in G.

<u>Def</u> ave. indep ratio $\frac{\bar{\alpha}_G(\lambda)}{|G|}$ occupancy fraction of G.

at fugacity λ.

<u>Thm</u> $\forall\, \lambda > 0$ and n-vx Δ-free G w./ $\Delta(G) \le d$

the occupancy fraction satisfies

$$\frac{1}{n} \bar{\alpha}_G(\lambda) \geq \frac{\lambda}{1+\lambda} \cdot \frac{W\left(d \log(1+\lambda)\right)}{d \log(1+\lambda)},$$

where for $z > 0$, $W(z)$ is the unique positive real

w./ $\quad W(z) \cdot e^{W(z)} = z$.

<u>Prop</u> The expected size $\bar{\alpha}_G(\lambda)$ is increasing in λ.

<u>Pf</u>: • Suffices to show $\bar{\alpha}_G'(\lambda) \geq 0$

$P = P_G(\lambda)$, $I \sim$ hard-core model

$$\mathbb{E}|I| = \bar{\alpha}_G(\lambda) = \frac{\lambda P'}{P}$$

$$\bar{\alpha}_G'(\lambda) = \left(\frac{\lambda P'}{P}\right)' = \frac{P'}{P} + \frac{\lambda P''}{P} - \frac{\lambda (P')^2}{P^2}$$

$$P'' = \sum |I|(|I|-1)\, \lambda^{|I|-2} = \frac{\mathbb{E}|I|^2 - \mathbb{E}|I|}{\lambda^2} \Rightarrow \quad = \cdots = \frac{\operatorname{Var}|I|}{\lambda} \geq 0$$

- To get a lower bound

$$\bar{\alpha}(G) \geq (1+o(1)) \frac{\log d}{d} \cdot n$$

$$\bar{\alpha}(G) = \bar{\alpha}_G(1) \underset{\lambda<1}{\geq} \bar{\alpha}_G(\lambda) =$$

Choose $\lambda = \frac{1}{\log d}$

<u>Idea</u>

$$(*) \quad \mathbb{E}|I| = \sum_{v \in V(G)} \Pr(v \in I) \quad \cdots \cdots \ 1$$

$$\mathbb{E}|I| = \bar{\alpha}_G(\lambda)$$

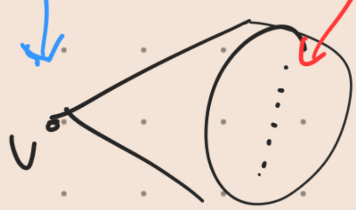
$$\geq \frac{1}{d} \sum_{v \in V(G)} \sum_{u \in N(v)} \Pr(u \in I) \quad \cdots \cdots \ 2$$

<u>Pf</u>: $I \sim$ hard-core model w/ fuga. λ.

- A vertex $v \in V(G)$ is suitable if

$N(v) \cap I = \phi$

none in I



so $v \in I$ only if v is suitable

- Counting in 1

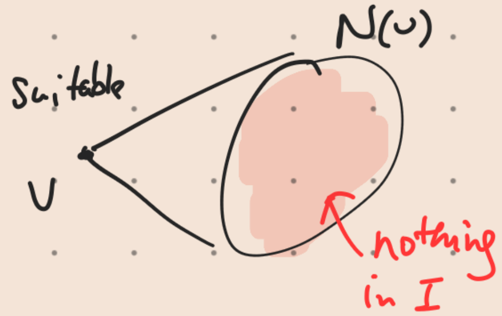
$$\Rightarrow \mathbb{E}|I| = \sum_{v \in V(G)} \Pr(v \in I) = \sum_{v} \Pr(v \in I \mid v \text{ suitable}) \cdot \Pr(v \text{ suitable})$$

<u>Claim</u> $\forall\, v \in V(G)$, $\Pr(v \in I \mid v \text{ suitable}) = \frac{\lambda}{1+\lambda}$.

<u>Pf</u>: Pair up choices of I by conditioning

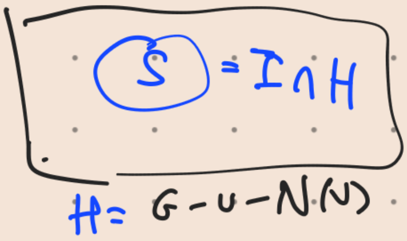
$I \cap (G - v - N(v)) = S$

two choices $\begin{cases} I = S \\ I = S \cup \{v\} \end{cases}$

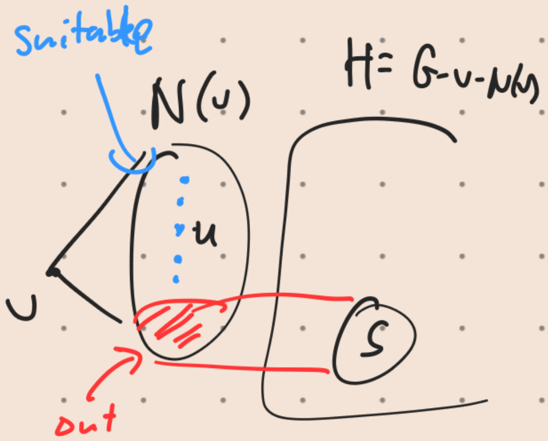


Pr. that $I = S \cup \{v\}$ is

$$\frac{\lambda^{|S|+1}}{\lambda^{|S|} + \lambda^{|S|+1}} = \frac{\lambda}{1+\lambda}$$



- Estimate $\Pr(v \text{ suitable})$



<u>Def</u>: X_v r.v. = # of suitable neighbors of v.

- so v is suitable when

NONE of the X_v suitable neighbors in I

- $N(v)$ = indep set $\Leftarrow \Delta$ freeness

$\forall u$ suitable in $N(v)$

$$\Pr(u \in I \mid u \text{ suitable}) = \frac{\lambda}{1+\lambda}$$

$$\Rightarrow \Pr(v \text{ suitable}) = \Pr\left(\text{none of } X_v \text{ suitable neighbors in } I\right)$$

$$= \sum_{x=0}^{d} \boxed{\left(\frac{1}{1+\lambda}\right)^x} \cdot \Pr(X_v = x).$$

↑ by independence