Lecture 19

Recall: • Γ finite abelian gp.

$A = -A \subseteq \Gamma \setminus \{0\}$ generating set

$Cay(\Gamma, A) = (V, E)$, $V = \Gamma$

$g, h \in \Gamma$, $g \sim h \iff g - h \in A$

$g = h + a$ for some $a \in A$

• characters of Γ:

gp homomorphism: $\Gamma \to \mathbb{C}^\times = S^1$

$\chi(a+b) = \chi(a)\chi(b)$

orthogonality: $\langle \chi_i, \chi_j \rangle = \sum_{g \in \Gamma} \chi_i(g) \cdot \chi_j(g) = 0$ $\forall i \neq j$

Rmk: It's conventional to define for functs: $\Gamma \to \mathbb{C}$

the inner prod. $\langle \chi, \gamma \rangle = \mathbb{E}_{g \in \Gamma} \chi(g) \overline{\gamma(g)}$.

$n = |\Gamma|$

$\chi_1, \dots, \chi_n$ form orthonormal basis of $\mathbb{C}^\Gamma$

funct: $\Gamma \to \mathbb{C}$.

• For $Cay(\Gamma, A)$: $\chi_1, \dots, \chi_i$ . . . . χ_n eig. vectors

$\updownarrow$

$\hat{A}(\chi_i) = \langle A, \chi_i \rangle$ eig. value

$= \sum_{a \in A} \chi_i(a)$

Chernoff type concentration

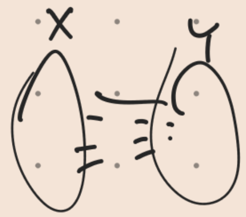
Lem 1 Let $p_1, \dots, p_n \in [0,1]$ and $\hat{p} = \frac{1}{n}\sum_{i=1}^{n} p_i$. Let X_i be centered Bernoulli r.v. i.e. $X_i := \begin{cases} 1-p_i & \text{w/ prob } p_i \\ -p_i & \text{prob. } 1-p_i \end{cases}$

and let $X = \sum_{i=1}^{n} X_i$. Then for any $a>0$,

$$\Pr(|X| > a) < 2e^{-2a^2/n}$$

Pf: $\underset{\text{WTS}}{=}$ $\neg EIG(\varepsilon) \Rightarrow \neg Disc(\delta)$.

NTF abnormal cut



$e(X,Y)$ deviates quite a bit from expected.

Let $d = |A|$ $\lambda_1 = d$, $\lambda_2 \geq \varepsilon d$

$G = Cay(\Gamma, A)$ d-reg

Supp for contradiction

$\neg EIG(\varepsilon)$ and $Disc(\delta)$.

We shall find such X, Y randomly by choosing ele^{t} $g \in \Gamma$ w./ prob. depending on $\chi \not\equiv 1$, a nontrivial character, whose corresponding eigenvalue is large:

$$|\hat{A}(\chi)| = |\langle A, \chi \rangle| \geq \varepsilon d.$$

- Write $\chi = c + i \cdot s = e^{i\theta_\chi}$ i.e. $\forall g \in \Gamma$

$c(g) = \mathrm{Re}(\chi(g)) = \cos(\theta_\chi)$, $s(g) = \mathrm{Im}(\chi(g)) = \sin(\theta_\chi)$

Orthog. of character $\chi \perp 1 \leftarrow$ trivial character

$$0 = \langle 1, \chi \rangle = \sum_{g \in \Gamma} \left(c(g) + i \cdot \underline{s(g)} \right)$$

$$\Rightarrow \sum_{g \in \Gamma} c(g) = 0, \quad \sum_{g \in \Gamma} s(g) = 0$$

A symm. $\Rightarrow \forall a \in A \;\; \exists -a \in A$

$$s(a) = -s(-a) \iff s(a) + s(-a) = 0$$

$$\Rightarrow \underline{\sum_{a \in A} s(a) = 0}$$

$$\langle A, \chi \rangle = \sum_{a \in A} \chi(a) = \sum_{a \in A} \left(c(a) + \cancel{i \cdot s(a)} \right)$$

$$= \langle A, c \rangle$$

- Set prob. vector $p = \frac{1+c}{2}$

Define X to be the random set obtained as follows: $\forall g \in \Gamma$, g is included in X w./ prob. $p(g) = \frac{1+c(g)}{2}$. indep. of other choices.

Def Y the same way.

$d = |A|$

- Goal is to bound deviation $\left| e(X,Y) - \frac{d}{n} |X||Y| \right|$

Let us first get a hold on sizes of X, Y.

$$g \rightsquigarrow p(g) = \frac{1+c(g)}{2}$$

$$\mathbb{E}|Y| = \mathbb{E}|X| = \sum_{g \in \Gamma} \frac{1+c(g)}{2} = n/2 \qquad n = |\Gamma|$$

By indep. & chernoff type Lem 1.

$$\mathbb{P}\left(|X| = \left(\tfrac{1}{2} + o(1)\right) n \right) = 1 - o(1)$$

$$\mathbb{P}\left(|Y| = \left(\tfrac{1}{2} + o(1)\right) n \right) = 1 - o(1)$$

By linearity of expectation,

$$\mathbb{E}\, |X \cap Y| = \sum_{g \in P} p(-g) \cdot p(g) = \sum_{g \in P} p(g)^2$$

$$= \frac{1}{4} \sum_{g \in P} (1 + c(g))^2 \underset{\langle 1, c \rangle = 0}{=} \frac{1}{4} n + \frac{1}{4} \sum_{g \in P} c(g)^2$$

$$m = \left| \{ \chi(g) : g \in P \} \right|$$

$$m = \begin{cases} 2 \\ > 2 \end{cases}$$

(calculation skipped for now)

$$= \frac{1}{4} n + \frac{1}{4} \cdot \frac{n}{2} = 3n/8$$

$$\sum_{g \in P} c(g)^2 = \begin{cases} n/2 & m > 2 \\ n & m = 2 \end{cases}$$

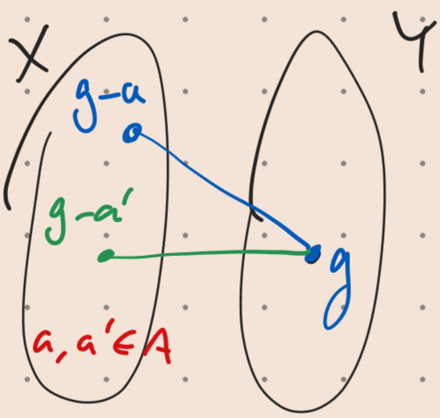
$$\text{Lem 1} \Rightarrow \mathbb{P}\left(|X \cap Y| = \left(\tfrac{3}{8} + o(1)\right) n \right) = 1 - o(1)$$

$$\mathbb{P}\left(|X \cup Y| = \left(\tfrac{5}{8} + o(1)\right) n \right) = 1 - o(1)$$

- $$\text{cut } e(X, Y) = \sum_{a \in A} \sum_{g \in P} X(g - a)\, Y(g)$$

$$= \sum_{a \in A} \underbrace{\left(\sum_{g \in P} (-X)(a - g) \cdot Y(g) \right)}$$

$$= \sum_{a \in A} (-X) * Y(a)$$



$$= \sum_{g\in\Gamma} A(g)\cdot(-X)*Y(g)$$

$$= \langle A, (-X)*Y\rangle = e(X,Y)$$

- $\mathbb{E}\, e(X,Y) = \mathbb{E}\langle A, (-X)*Y\rangle$

$$= \sum_{a\in A}\sum_{g\in\Gamma} \mathbb{E}(-X)(a-g)\,\mathbb{E}\,Y(g)$$

lin. exp. & also indep. of X, Y

$$= \sum_{a\in A}\left(\sum_{g\in\Gamma} p(a-g)\cdot p(g)\right) = \sum_{a\in A} p*p(a)$$

$$= \sum_{a\in A} \tfrac{1}{2}(1+c)*\tfrac{1}{2}(1+c)\,(a)$$

$$= \left\langle A, \tfrac{1}{2}(1+c)*\tfrac{1}{2}(1+c)\right\rangle$$

$$= \frac{1}{4}\sum_{a\in A}\sum_{g\in\Gamma}\big(1+c(a-g)\big)\cdot\big(1+c(g)\big)$$

$$= \frac{1}{4}\sum_{a\in A}\sum_{g\in\Gamma}\big(1+c(a-g)\,c(g)\big) \qquad \langle 1, c\rangle = 0$$

$$= \frac{1}{4}dn + \frac{1}{4}\langle A, c*c\rangle$$

Claim: $c*c\,(g) = \begin{cases} \frac{n}{2}c(g) & m>2 \\ n\cdot c(g) & m=2 \end{cases}$.

$\frac{|A|}{n}$ $n/2$ $n/2$

$\underbrace{\frac{d}{n}|X||Y|}$

$$\langle A, c*c\rangle = \sum_{g\in P} A(g)\cdot c*c(g)$$

$$\overset{k>2}{=} \sum_{a\in A} \frac{n}{2} c(a) = \frac{n}{2}\langle A, c\rangle$$

calculation

$$\underline{\left|\mathbb{E}\, e(X,Y) - \frac{1}{4}dn\right|} \geq \frac{1}{8} n |\langle A, c\rangle|$$

$$\underline{\geq \frac{1}{8} n\cdot \varepsilon d.} \qquad (*)$$

Recall $0 \le e(X,Y) \le dn = |A|n$.

$$\eta = \eta(X,Y) = e(X,Y) - \frac{1}{4}dn$$

$$\Rightarrow \quad -\frac{1}{4}dn \le \eta \le \frac{3}{4}dn$$

bad event

Write $q = \Pr\left(|\eta| \le \frac{\varepsilon dn}{16}\right)$

$$\underline{\frac{1}{8}\varepsilon dn \overset{(*)}{\le} |\mathbb{E}(\eta)|} \overset{\Delta\text{-ineq}}{\le} \mathbb{E}(|\eta|)$$

$$\le q\cdot \frac{\varepsilon dn}{16} + (1-q)\frac{3}{4}dn$$

$$\Rightarrow \quad q = \Pr\left(|\eta| \le \frac{1}{16}\varepsilon dn\right) \le \frac{1-\varepsilon/6}{1-\varepsilon/12} \le 1 - \frac{\varepsilon}{12}$$

So $\exists$ choices of X, Y s.t. not happening

& $|X|, |Y|$ is as expected

That is ; $|X| = |Y| = \left(\frac{1}{2} + o(1)\right) n$

$\boxed{n > 2}$ $|X \cap Y| = \left(3/8 + o(1)\right) n$

$|X \cup Y| = \left(5/8 + o(1)\right) n$

$|\eta| = |e(X,Y) - \frac{1}{4} d n| \geq \frac{1}{16} \varepsilon d n$.

Apply $\underline{\text{Disc}(\delta)}$ on $X \cup Y$, $X \setminus Y$, $Y \setminus X$, $X \cap Y$

$$\Rightarrow e(X,Y) = e(X \cup Y) - e(X \setminus Y) - e(Y \setminus X) + e(X \cap Y)$$

$$\frac{1}{16}\varepsilon d n \leq \left| e(X,Y) - \frac{1}{4} d n \right| < \cancel{\frac{40}{128}} \frac{5}{16} \delta d n$$

Setting $\delta < \frac{\varepsilon}{5}$ we get a contradiction

$\blacksquare$