Lecture 18

We proved:

$$\begin{tikzcd}
\text{[Ind. subg. count]} \arrow[r, Rightarrow] & \text{[subg. count]} \arrow[r, Rightarrow] & \text{[}C_4\text{ count]} \arrow[r, Rightarrow] & \text{[Spectral gap]} \arrow[d, Rightarrow] \\
 & & \text{[Codegree]} \arrow[ull, Rightarrow] & \text{[Discrepancy]} \arrow[l, Rightarrow]
\end{tikzcd}$$

Rmk: In retrospect, it is perhaps not that surprising how that the seemingly weaker property of [C_4 count] is equiv. to [Ind. subg. count].

Indeed, we've seen that [C_4] $\Rightarrow$ [Codegree]

In the pf of [codeg] $\Rightarrow$ [Ind. subg. count], we count H-subgraphs by building it up one vertex at a time. $H_1, H_2, \ldots, H_r, \ldots, H_s = H$. Let H^*_{r+1} be the graph obtained from H_{r+1} by adding a new vertex v^*_{r+1}, which is a copy of v_{r+1}. To count H_{r+1}, we need to control the 'variance': the # of H^*_{r+1}, which in view of the twins v_{r+1} and v^*_{r+1}, is governed by the [codeg] property.

Consider sparse graphs and see what still holds.

$d = o(n)$

Expander mixing Lem ☺

$$[\text{EIG}] \xrightarrow{\text{Expander mixing Lem}} [\text{Disc}]$$

no longer true for sparse graphs

Spectral gap

$|\lambda_2| = o(d)$

Discrepancy

$$\left| e(A,B) - \frac{d}{n}|A||B| \right| = o(dn)$$

Ex

d-reg random graph

$n-d-1$ vxs

K_{d+1}

- d-reg
- [Disc]

But $\lambda_1 = \lambda_2 = d$ no [EIG].

What if we impose more symmetry to exclude such examples?

Consider d-reg graphs

Thm [Disc $\Rightarrow$ EIG for all Cayley graphs]

[Kohayakawa - Rödl - Schacht]

Def: [Disc(δ)] Let $0 < \delta \le 1$. We say a d-reg graph G satisfies Disc(δ) if $\forall\, U, V$ disjoint subsets of $V(G)$, we have

$$e_G(U,V) = (1 \pm \delta)\frac{d}{n}|U||V|$$

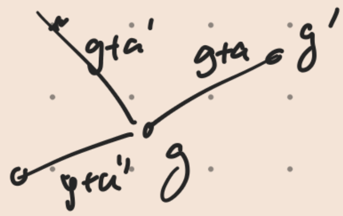
Def: [EIG(ε)] $\overset{d}{\overset{\shortparallel}{\lambda_1}} \geq \lambda_2 \geq \dots \geq \lambda_n$ e.g. of adj mat. of G. $\forall i \geq 2$ $|\lambda_i| \le \varepsilon \cdot d$

Def: (Cayley graph) Let Γ be an abelian group and let $A \subseteq \Gamma \setminus \{0\}$ be symmetric.
(i.e. if $a \in A \Rightarrow -a \in A$)
$A = -A$.

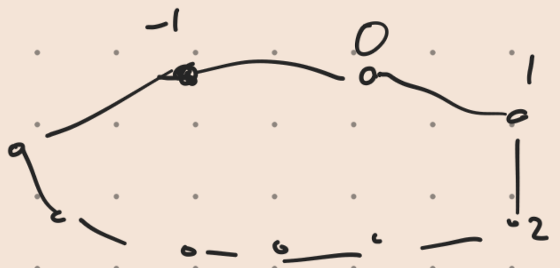
generating set

The Cayley graph $\mathrm{Cay}(\Gamma, A)$ is the graph with vertex set $= \Gamma$ and two vertices g, g' form an edge iff $g - g' \in A$

Ex $\Gamma = \mathbb{Z}_N$, $A = \{1, -1\}$



$\mathrm{Cay}(\mathbb{Z}_N, A) = C_N$ N-cycle



vertex-transitive

We shall see a pf by Gowers.

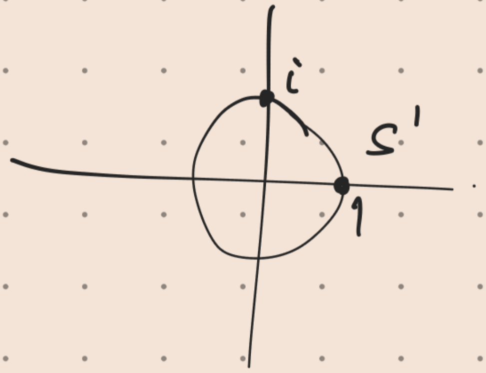
Thm [K-R-S/G] $\forall \varepsilon > 0$, $\exists \delta > 0$ and n_0 s.t. the following holds. Let $G = \mathrm{Cay}(\Gamma, A)$ be a Cayley graph for some abelian gp Γ with $|\Gamma| = n \geq n_0$ and a symmetric $A = -A \subseteq \Gamma \setminus \{0\}$.
Then $\mathrm{DISC}(\delta) \Longrightarrow \mathrm{EIG}(\varepsilon)$.

§. Basics on Caylay graphs

Let Γ be a finite abelian gp.

A **character** χ of Γ is a group homomorphism from $\Gamma \to S^1$, where S^1 is the multiplicative group of complex numbers

$$\chi : \Gamma \to S^1$$



$\forall\, a,b \in \Gamma$, $\chi(a+b) = \chi(a)\cdot\chi(b)$

Ex All 1 funct is a trivial character.

If $|\Gamma| = n$, then there are n characters $\chi_1, \dots, \chi_n$ pairwise orthogonal, i.e. (view χ_i as vector of length n taking $\mathbb{C}$-value)

$$\langle \chi_i, \chi_j \rangle = \sum_{g\in\Gamma} \chi_i(g)\, \chi_j(g)$$

- Consider the space of funct. from Γ to $\mathbb{C}$, the the characters $\chi_1,\dots,\chi_n$ form an orthonormal basis of this space.

$\forall\, \chi$ ch. $\forall g \in \Gamma$

- $\chi(g)$ is m-th root of unity, where m is the order of g in Γ.

Equiv., if $|\{\chi(g) : g \in \Gamma\}| = m$, then χ takes values in m-th root of unity.

- Eigenvalues of Cayley graphs $\mathrm{Cay}(\Gamma, A)$ are Fourier coeff. of A (view as a characteristic funct)

<u>Thm</u> [Lovász 75]

Let $G = \mathrm{Cay}(\Gamma, A)$ w/ Γ finite abelian and symm. $A = -A \subseteq \Gamma \setminus \{0\}$. Then every character χ is an eigenvector of the adj. mat. of G w/ eigenvalue

$$\hat{A}(\chi) = \langle A, \chi \rangle = \sum_{a \in A} \chi(a).$$

<u>Pf</u>: Let M be the adj. mat. of G.

<u>NTS</u>: $$M \cdot \chi = \langle A, \chi \rangle \cdot \chi = \left(\sum_{a \in A} \chi(a)\right) \cdot \chi$$

Fix a coord. $g \in \Gamma$.

$$(M \cdot \chi)_g = \sum_{b \in N(g) = g + A} \chi(b)$$

$$\begin{array}{cc} & \begin{array}{ccc} g+a & M & g+a' \end{array} \\ g & \begin{pmatrix} 1 & 0 & 1 \end{pmatrix} \end{array} \quad \begin{array}{c} \chi \\ \begin{pmatrix} \vdots \end{pmatrix} \end{array}$$

$N(g) = g + A = g - A$

$$= \sum_{a \in A} \chi(g + a)$$

$$= \sum_{a \in A} \chi(g) \cdot \chi(a) = \left(\sum_{a \in A} \chi(a)\right) \cdot \chi(g) \qquad \blacksquare$$

Def: Given two funct. f and $h : \Gamma \to \mathbb{C}$

let $f * h : \Gamma \to \mathbb{C}$ be their convolution

defined as, $\forall\, a \in \Gamma$

$$f * h(a) = \sum_{g \in \Gamma} f(a-g)\, h(g)$$

Idea of the pf $[\mathrm{Disc}(\delta)] \Rightarrow [\mathrm{EIG}(\varepsilon)]$.

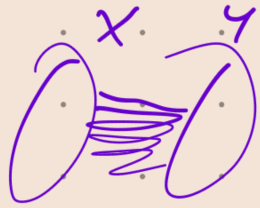
Want to prove the contrapositive that

$$\neg[\mathrm{EIG}(\varepsilon)] \Longrightarrow \neg[\mathrm{Disc}(\delta)]$$

$\lambda_1 = d$, $\lambda_2 \geq \varepsilon d$

$= |A|$

need to find two sets X & Y s.t. $e(X,Y)$ is abnormal.



Idea: Look at χ_2 (eig vector of λ_2), define random sets X, Y using χ_2. Place $g \in \Gamma$ in X w./ prob p depending on $\chi_2(g)$ indep. of other elements.

$A \subseteq \Gamma$, $A(g) = \begin{cases} 1 & \text{if } g \in \Gamma \\ 0 & \text{o.w.} \end{cases}$

Ex: $\mathrm{supp}(A * A) \supseteq A + A$