Lecture 17

Today: [Spectral gap] $\Longrightarrow$ [Discrepancy] $\Longrightarrow$ [codegree]

$|\lambda_2| = o(n)$ $\qquad$ $e(A,B) = p|A||B| + o(n^2)$ $\qquad$ $\sum_{u,v} |d(u,v) - p^2 n| = o(n^3)$

$\lambda_1(G) = d$ ← exer.

Def: ((n,d,λ)-graph) An n-vx d-regular graph G is an (n,d,λ)-graph if all nontrivial eigenvalues have absolute value at most λ.

Expander mixing lemma

Let G be an (n,d,λ)-graph. Then for any $S,T \subseteq V(G)$ we have

$$\left| e(S,T) - \overset{=p}{\frac{d}{n}}|S||T| \right| \le \lambda \sqrt{|S||T|}.$$

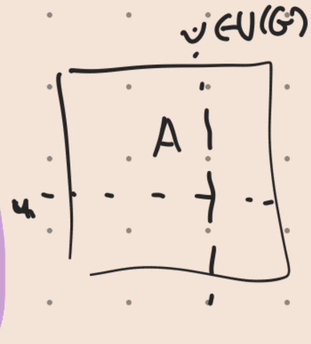
Pf: • Let A be the adj. matrix of G w/ eig value/vect. $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$

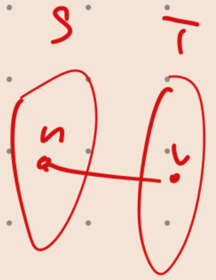
$\mathbf{1} = v_1 \quad v_2 \quad \cdots \quad v_n$

• $|\lambda_i| \le \lambda \quad \forall\, i \geq 2$.

• Use S for its characteristic funct. $S(x) = \begin{cases} 1 & x \in S \\ 0 & x \notin S \end{cases}$

$$e(S,T) = S^T A T = \sum_{u,v} S(u) A_{u,v} T(v)$$





$$\boxed{\begin{aligned} S &= \sum_{i=1}^{n} \alpha_{v_i} \cdot v_i \\ T &= \sum_{i=1}^{n} \beta_{v_i} \cdot v_i \end{aligned}}$$

$\langle v_i, v_j \rangle = 0$

$$\boxed{\boxed{\langle S, v_1 \rangle = \alpha_{v_1} \cdot v_1^T v_1} \quad \lambda_i v_i}$$

$$AT = A \sum \beta_{v_i} v_i = \sum \beta_{v_i} A v_i$$

$$= \sum \lambda_i \beta_{v_i} \cdot v_i$$

$$\underbrace{S^T}_{\text{III}}$$

$$\left\langle \sum_{i=1}^{n} \alpha_{v_i} \cdot v_i , \sum_{i=1}^{n} \lambda_i \beta_{v_i} \cdot v_i \right\rangle$$

$$= \sum_{i=1}^{n} \lambda_i \cdot \alpha_{v_i} \cdot \beta_{v_i} \; v_i^T v_i$$

$$= \underline{d \cdot \alpha_{v_1} \beta_{v_1} \cdot 1^T \cdot 1} + \sum_{i=2}^{n} \lambda_i \alpha_{v_i} \beta_{v_i} v_i^T v_i$$

$\alpha_{v_1} = \langle S, v_1 \rangle$ $\quad \cdot \; \alpha_{v_1} = \dfrac{\langle S, v_1 \rangle}{v_1^T v_1}$

$$d \cdot \alpha_{v_1} \beta_{v_1} \cdot v_1^T v_1 = d \underbrace{\langle S, v_1 \rangle}_{|S| \, v_1} \underbrace{\langle T, v_1 \rangle}_{|T|} \cancel{\underbrace{v_1^T v_1}_{n}}$$

$$= \frac{d}{n} \cdot |S| \, |T|$$

$$e(S,T) - \frac{d}{n} |S||T| = \sum_{i=2}^{n} \lambda_i \alpha_{v_i} \beta_{v_i} v_i^T v_i$$

$$\left| \; \ldots \; \right| \le \underbrace{\sum_{i=2}^{n}} |\lambda_i| \, \underbrace{\left| \alpha_{v_i} \beta_{v_i} v_i^T v_i \right|}$$

$$\le \lambda \sqrt{|S||T|}$$

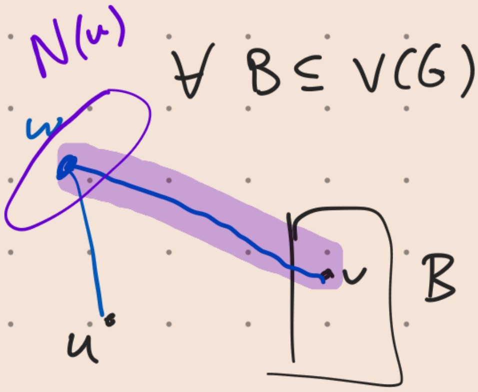
Ex Cauchy-Schwarz

[Discrepancy $\Rightarrow$ Codegree]

<u>Will show:</u> $\forall\ u \in V(G)$ $$\sum_{v:\, v \neq u} \left| d(u,v) - p^2 n \right| = o(n^2)$$

Express codegree as edges between $N(u)$ and some set.

$$\forall\ B \subseteq V(G), \quad \sum_{v \in B} \left(d(u,v) - p^2 n \right) = \sum_{v \in B} d(u,v) - |B| p^2 n$$



$$= e(N(u), B) - |B| p^2 n$$

To get rid of the abs. value sign,

define $V(G) \setminus \{u\} = B^+ \cup B^-$, where

$$B^+ := \{ v : d(u,v) > p^2 n \}$$
$$B^- := \{ v : d(u,v) \le p^2 n \}$$

$$\sum_{v:\, v \neq u} \left| d(u,v) - p^2 n \right| = \sum_{v \in B^+} \left(d(u,v) - p^2 n \right) - \sum_{v \in B^-} \left(d(u,v) - p^2 n \right)$$

$A = N(u)$

$|A| = pn$

$$= \underbrace{e(A, B^+) - p^2 n \cdot |B^+|}_{p \cdot |A| \cdot |B^+|} - \left[\underbrace{e(A, B^-) - p^2 n |B^-|}_{p|A||B^-|} \right]$$

$$\overset{[\text{Disc}]}{=} o(n^2)$$

∎

Q: What about sparser graphs (edge density $\to 0$ as $n \to \infty$)?

$|\lambda_2| = o(n) \ll \lambda_1 = p \cdot n$

$$[\text{Spectral gap}] \xrightarrow{\text{expander mixing lem}} [\text{Discrepancy}]$$

not true

$|\lambda_2| = o(d)$

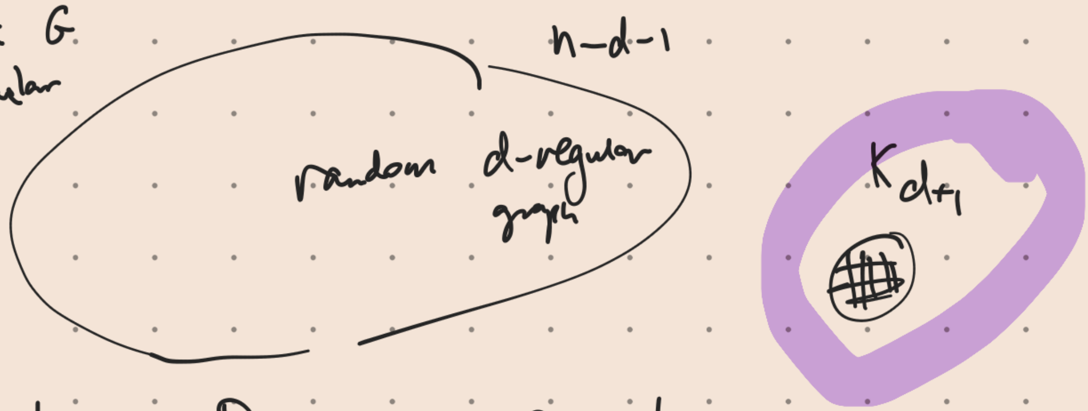
$\left| e(A,B) - \frac{d}{n}|A||B| \right| = o(n^2)$ $\quad = \lambda\sqrt{|S||T|}$

$o(d \cdot n)$

For sparse graphs [Spe. gap] still $\Rightarrow$ [Disc].

but not necessarily the other way around.

Ex: n-vx G
d-regular
$d = o(n)$



G has Discrepancy property but not having spectral gap.

Ex: For a d-regular graph G, the multiplicity of the eig. value d is exactly the # connected components of G.

Next week

Thm [Kohayakawa - Rödl - Schacht / Gowers]

[Disc] & (Spe gap) are equivalent for Cayley graphs.