Lecture 16

Quasirandom properties

d-reg, $d = p(n-1)$ $\qquad t_{ind}(H,G)$, $t(H,G)$

- [Induced subgraph count] $\forall H$

Ex. $\qquad t_{ind}(H,G) = p^{e(H)}(1-p)^{e(\bar{H})} + o(1)$

- [Subgraph count] $\forall H$. $t(H,G) = p^{e(H)} + o(1)$

Defn.

?

- [4-cycle count] $\quad t(C_4, G) \le p^4 + o(1)$
- [Spectral gap] $\quad |\lambda_2| = o(n)$ $\qquad |\lambda_1| \ge |\lambda_2| \ge \cdots \ge |\lambda_n|$

Expander mixing lem

- $\Rightarrow$ [Discrepancy] $\forall A, B \subseteq V(G)$, $e(A,B) = p|A||B| + o(n^2)$

?

- [Codegree] $\displaystyle\sum_{u,v} |d(u,v) - p^2 n| = o(n^3)$.

Rmk (i), $e(A,B) = \#\underbrace{(u,v)}$ ordered pairs s.t. $\begin{cases} uv \in E(G) \\ u \in A \quad v \in B \end{cases}$

e.g. if $A = B$ $\quad e(A,A) = 2e_G(A)$

- When we say Spectral gap implies Discrepancy:

More precisely: $\quad \delta, \varepsilon > 0$

$\underline{EIG(\delta)}$: $\quad |\lambda_2| \le \delta \cdot n$

$\underline{DISC(\varepsilon)}$: $\quad e(A,B) = p|A||B| + \varepsilon n^2$

$\forall \varepsilon > 0$, $\exists \delta > 0$ and $n_0 \in \mathbb{N}$ s.t. the following holds for $\forall n \ge n_0$. Let G be an n-vx graph

if G satisfies $EIG(\delta) \Rightarrow G$ also satisfies $DISC(\varepsilon)$.

Today: [codeg] $\Rightarrow$ [induced subgr. counts]

- $V(H) = \{v_1, v_2, \dots, v_s\}$
- for $r \in [s]$, define $H_r = H[\{v_1, \dots, v_r\}]$

$$H = H_s$$

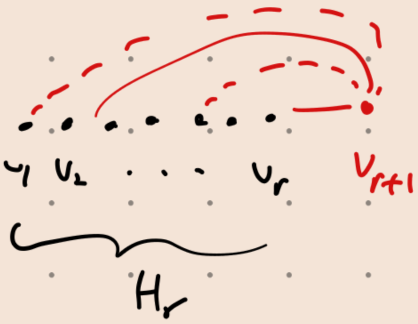
Shall use induction on $1 \le r \le s$, via building H_{r+1} from H_r to show that G has the 'correct' count of induced copies of $H = H_s$

Let N_r = # labelled induced copies of H_r.

Induction: to show $N_r = (1+o(1))\, n^r p^{e(H_r)} (1-p)^{e(\overline{H_r})}$ ..... ☺

- Base case $r = 1$ trivially true.
- Assume true for $1 \le r < s$, will show ☺ for $r+1$.

- <u>Extension function</u>: Let $\varepsilon \in \{0,1\}^r$ recording the adjacencies of v_{r+1} to H_r in H_{r+1},



i.e. $\forall j \in [r]$ $\varepsilon_j = 1$ iff $v_j v_{r+1} \in E(H_{r+1})$

Notation $V^{(r)}$ = ordered r-tuples in V, $|V^{(r)}| = n_{(r)}$

$\forall \ \omega \in V^{(r)}$, define

$$X(\omega) = \left| \left\{ v \in V(G) : v \notin \omega \text{ and } v \sim w_j \iff \xi_j = 1 \ \ \forall j \in [r] \right\} \right|$$

It's convenient to view things probabilistically.

Let $\Omega = V^{(r)}$, $\Omega^* := \{ \omega \in \Omega : \omega \cong H_r \}$

$$\Rightarrow \quad N_{r+1} = \sum_{\omega \in \Omega^*} X(\omega)$$

Endow Ω w/ unif prob. measure and X r.v.

$$\forall \ \omega \in \Omega, \quad \Pr(X = X(\omega)) = \frac{1}{n_{(r)}}.$$

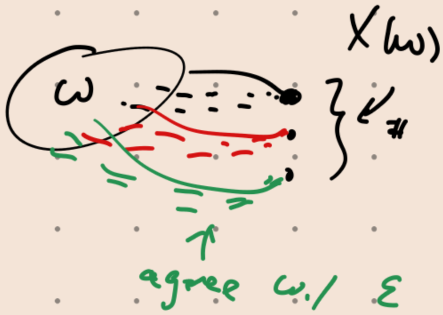
- Concentration of X:

$$\boxed{(\heartsuit) \cdots \quad \sum_{\omega \in \Omega^*} X(\omega) = \mathbb{E} X \cdot |\Omega^*| + o(n^{r+1})}$$

Assuming $(\heartsuit)$ for now,

$$\boxed{(\mathrm{IH}) \Rightarrow |\Omega^*| = N_r = (1+o(1))\, n^r p^{e(H_r)} (1-p)^{e(H_r)}}.$$

$$\mathbb{E} X = \frac{1}{|\Omega|} \sum_{\omega \in \Omega} X(\omega)$$



We count $\sum_{\omega \in \Omega} X(\omega)$ from the perspective of the $(r+1)^{th}$-vx.

$\forall \ u_{r+1}$, #ω's attaching to u_{r+1} agreeing w./ ε.

$|\varepsilon| = \#\ 1\text{s in } \varepsilon \in \{0,1\}^r$

$$= (pn)^{|\varepsilon|}\left[(1-p)n\right]^{r-|\varepsilon|}$$

$$= p^{|\varepsilon|}(1-p)^{r-|\varepsilon|}\, n^r .$$

$$\Rightarrow \sum_{\omega \in \Omega} X(\omega) = (1+o(1))\, p^{|\varepsilon|}(1-p)^{r-|\varepsilon|}\, n^{r+1}$$

$$\Rightarrow \mathbb{E}X = \frac{1}{|\Omega|}\sum_{\omega\in\Omega} X(\omega) = (1+o(1))\, p^{|\varepsilon|}(1-p)^{r-|\varepsilon|}\, n$$

N_r by IH

$$N_{r+1} = \sum_{\omega\in\Omega^*} X(\omega) \overset{(\heartsuit)}{=} \mathbb{E}X \cdot |\Omega^*| + o(n^{r+1})$$

$$= (1+o(1))\, p^{|\varepsilon|}(1-p)^{r-|\varepsilon|}\, n \cdot p^{e(H_r)}(1-p)^{e(\overline{H_r})} \cdot n^r + o(n^{r+1})$$

$|\varepsilon| + e(H_r) = e(H_{r+1})$

$r - |\varepsilon| + e(\overline{H_r}) = e(\overline{H_{r+1}})$

$$= (1+o(1))\, p^{e(H_{r+1})}(1-p)^{e(\overline{H_{r+1}})}\, n^{r+1}$$

- We are left to show

$$(\heartsuit) \quad \sum_{\omega\in\Omega^*} X(\omega) = \mathbb{E}X \cdot |\Omega^*| \boxed{+\, o(n^{r+1})}.$$

Exer. Use Cauchy-Schwartz to show the following:

Lem Let X be a r.v. over finite set Ω w/ unif meas.

Let $\Omega^* \subseteq \Omega$

$$\Rightarrow \sum_{\omega\in\Omega^*} X(\omega) = \mathbb{E}X \cdot |\Omega^*| \boxed{\pm \sqrt{|\Omega|\cdot|\Omega^*|\cdot \mathrm{Var}(X)}}.$$

Pf of (♡) $\quad \mathrm{Var}(X) = \mathbb{E}(X-\mathbb{E}X)^2 = \cancel{\frac{1}{|\Omega|}} \sum_{\omega\in\Omega} (X-\mathbb{E}X)^2$

$$\sqrt{\cancel{|\Omega|}\,|\Omega^*|\,\mathrm{Var}(X)} = \sqrt{|\Omega^*| \cdot \sum_{\omega\in\Omega} (X-\mathbb{E}X)^2}$$

$$= \sqrt{|\Omega^*| \sum_{\omega\in\Omega} \left(X(\omega)^2 - (\mathbb{E}X)^2\right)}$$

$$\overset{\text{WTS}}{=} o(n^{r+1})$$

Recall $|\Omega^*| \overset{(IH)}{=} N_r = O(n^r)$

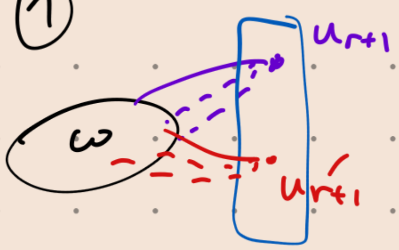
$\Rightarrow$ suffices to show

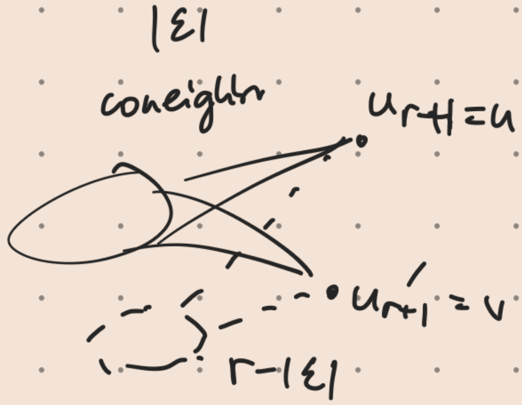
$$\sum_{\omega\in\Omega} X(\omega)^2 = |\Omega|\cdot(\mathbb{E}X)^2 + o(n^{r+2})$$

$$= p^{2|\varepsilon|}(1-p)^{2(r-|\varepsilon|)}\, n^{r+2} + o(n^{r+2})$$

We shall approx it by

$$T = \sum_{\omega\in\Omega} X(\omega)\left[X(\omega)-1\right].$$

① 

② 

Counting from ②'s perspective

$T = \#\,(\omega, \overset{\text{ordered pair}}{\{u,v\}})$ s.t. both u and v attach to ω as ε.

$$T = \sum d_G(u,v)^{|\mathcal{E}|}\, d_{\bar{G}}(u,v)^{r-|\mathcal{E}|}$$

<u>Claim</u> $\forall\, u \neq v \in V \quad \forall\, k, k' \geq 1$ integers

$$\Rightarrow \sum_{u \neq v} d_G(u,v)^k\, d_{\bar{G}}(u,v)^{k'} = (1+o(1))\, p^{2k} (1-p)^{2k'} n^{k+k'+2}.$$

Claim $\Rightarrow$ $T = (1+o(1))\, p^{2|\mathcal{E}|} (1-p)^{2(r-|\mathcal{E}|)} n^{r+2}$

$$\sum_{w \in \Omega} X(w)^2 = T + \sum_{w \in \Omega} X(w) = T + O(n^{r+1})$$

$$= (1+o(1))\, p^{2|\mathcal{E}|} (1-p)^{2(r-|\mathcal{E}|)} n^{r+2} \quad \text{as desired.}$$

$\blacksquare$

<u>Ex</u> prove the above Claim.

Hint: • Let $\delta_{uv} = d_G(u,v) - p^2 n$

[Codeg] $\Rightarrow \forall\, k \in \mathbb{N}, \quad \sum_{u \neq v} |\delta_{uv}|^k = o(n^{k+2})$

• $\bar{\delta}_{uv} = d_{\bar{G}}(u,v) - (1-p)^2 n$

$$\sum_{u \neq v} |\bar{\delta}_{uv}|^k = o(n^{k+2})$$